Certain Problems on the Absolute Summability Methods Based on Power Series

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I further certify that the work of this thesis, either partly or fully, has not been submitted to any other institution for the award of any other degree.

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CERTAIN PROBLEMS ON THE ABSOLUTE SUMMABILITY METHODS BASED ON POWER SERIES

SUMMARY

The present Thesis consists of eight chapters.

Chapter I is introductory, which contains a brief résumé of relevant results which have direct interconnections with our investigations. Definitions of some of the special summability methods are given after introducing the idea of absolute summability in general.

Let Σa_n be a given infinite series and let $\{s_n\}$ be the sequence of its partial sums, or any sequence of real or complex numbers such that $s_o \neq 0$.

Suppose that $p_n > 0$, $\sum_{n=o}^{\infty} p_n = \infty$, and that the radius of convergence of the power series

$$p(x) = \sum_{n=o}^{\infty} p_n x^n \; ; \quad p(0) = p_o ,$$

is 1. We shall use the notations :

$$(*) \qquad p_s(x) = \sum_{n=o}^{\infty} p_n s_n x^n ,$$

and

$$J(x) = p_s(x) / p(x).$$

If the series on the right of $(*)$ is convergent in the right open interval $(0, 1)$ and if

$$J(x) \in BV\,(c, 1)\ ,\ (0 < c < 1),$$

we say that the series $\sum_{n=0}^{\infty} a_n$, or the sequence $\{s_n\}$, is absolutely summable (J, p_n), or simply summable $|J, p_n|$.

We shall say that the series $\sum_{n=0}^{\infty} a_n$, is summable $|J, p_n|_k$, $k \geq 1$, if the series on the right of $(*)$ is convergent for $0 \leq x < 1$, and if

$$\int_c^1 (1-x)^{k-1} \left|\frac{d}{dx}\{J(x)\}\right|^k dx < \infty,\quad 0 < c < 1 .$$

In the special cases in which $p_n = \binom{n+\alpha}{n}$, $\alpha > -1$, and $p_n = \frac{1}{n+1}$, for $n=0,1,2,\ldots$, the methods $|J, p_n|$ and $|J, p_n|_k$, $k \geq 1$, reduce respectively to $|A_\alpha|$, $|A_\alpha|_k$ and $|L|$, $|L|_k$-methods. $|A_0|$ and $|A_0|_k$ are the same as the absolute Abel and generalized absolute Abel methods.

Let $\{s_n^\lambda\}$ be the sequence of associated (C, λ)-means of the sequence $\{s_n\}$ and let us write

$$(**)\qquad g_\lambda(x) = (1-x)\sum_{n=0}^{\infty} s_n^\lambda x^n,\quad (\lambda > -1;\ 0 \leq x < 1).$$

If the series on the right of $(**)$ converges for $0 \leq x < 1$ and if

$$g_\lambda(x) \in BV\,(\delta,\,1)\,, \quad 0 < \delta < 1\,,$$

we say that $\{s_n\}$ is absolutely summable $(A,\ \lambda)$, or simply summable $|A,\ \lambda|$.

We write

$$P_n = p_0 + p_1 + \cdots + p_n, \quad P_{-1} = p_{-1} = 0, \quad n=0,1,2,\ldots,$$

and

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v\, s_v \qquad (\,n \geq 0\,),$$

with $p_n > 0$. If $\{\sigma_n\} \in BV$, we say that the series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is absolutely summable $(\bar{N},\ p_n)$, or simply summable $|\bar{N},\ p_n|$.

We also write

$$t_n = (P_n)^{-1} \sum_{v=1}^{n} P_{v-1}\, a_v\,, \quad t_0 = 0\,.$$

The sequence-to-sequence quasi-Hausdorff transform, or simply the (H^*,χ)-transform, h_n^*, of the sequence $\{s_n\}$ is defined by :

$$h_n^* = \sum_{k=n}^{\infty} \int_0^1 s_k \binom{k}{n} t^{n+1} (1-t)^{k-n}\, d\chi(t), \quad (n=0,1,2,\ldots),$$

where $\chi(t)$ is a function of bounded variation in the closed interval $[0, 1]$. The sequence $\{s_n\}$ is said to be absolutely summable (H^*, χ), or simply summable $|H^*, \chi|$, if $\{h^*_n\} \in BV$.

The Chapter II has been devoted to the study of some of the properties of $|L|$-method and to consider its relationship to $|A, \lambda|$. Following theorems are proved :

Theorem 1. The $|L|$-method is translative. By this we mean $s_{n+1} \to s\ |L|$ if and only if $s_n \to s\ |L|$.

Theorem 2. For $\lambda > -1,\ \delta > 0,\ |A, \lambda| \subset |A, \lambda+\delta|$.

Theorem 3. Let $\lambda > -1,\ \delta > 0$. Then there is an $|A, \lambda+\delta|$-summable sequence which is not $|A, \lambda|$-summable.

Theorem 4. For $-1 < \lambda \leq 1,\ |A, \lambda| \subset |L|$.

Theorem 5. There is an $|L|$-summable sequence which is not $|A, \lambda|$-summable for $-1 < \lambda \leq 1$.

We see that these theorems are the analogues of certain results of BORWEIN [1)] and also extend a number of results previously proved by AHMAD. [2)]

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1) Borwein, D. : Jour. London Math. Soc., 33(1958), 212-220.

2) Ahmad, Z.U. : Ph.D. Thesis, University of Jabalpur, 1967; Alig. Bull. Math., 1(1971), 31-37; Rend. Mat. (6), 5(1972), 541-549.

In Chapter III, we prove the following theorem for b-translativity of the method $(J,\ p_n)$.

<u>Theorem 6</u>. <u>The</u> $(J,\ p_n)$<u>-method is b-translative if, and only if,</u> $\sum_{n=0}^{\infty} |\Delta p_n| \leq K$ <u>and</u> $\sum_{n=0}^{\infty} p_n = \infty$.

We get the following known results from our theorem.

<u>Corollary I</u>. <u>The method</u> (A_k), $-1 < k \leq 1$, <u>is b-translative</u>. <u>In particular, Abel method is b-translative</u>.

<u>Corollary II</u>. <u>The</u> (L)<u>-method is b-translative</u>.

In Chapter IV, we obtain three Tauberian theorems for $|J,\ p_n|$-summability and we observe that one of these includes, as a special case, the Tauberian theorem of HYSLOP [1] for absolute Abel summability.

<u>Theorem 7</u>. <u>If</u> $\Sigma\, a_n$ <u>is summable</u> $|J,\ p_n|$, <u>and</u> $\{t_n\} \in BV$, <u>and if</u> $\{p_n\}$ <u>is such that</u>

$$\text{(i)} \qquad n\, p_n / P_{n-1} < C, \quad \text{for } n=1,2,\ldots,$$

and

$$\text{(ii)} \quad N(w) = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u - v) P_u\, P_{v-u}}{\displaystyle\sum_{v=0}^{\infty} e^{-v/w} \sum_{u=0}^{v} (v-u+1) P_u\, P_{v-u}} > C \text{ for } w \geq 1,$$

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1) Hyslop, J.M. : Jour.London Math.Soc.,12(1937),176-180.

where C *is a strictly positive constant, then* Σa_n *is summable* $|\bar{N}, p_n|$.

Theorem 8. If Σa_n *is summable* $|J, p_n|$, *and* $\{t_n\} \in BV$, *and if* $\{p_n\}$ *satisfies the same conditions as in Theorem* 7, *then* Σa_n *is absolutely convergent.*

Theorem 9. If Σa_n *is summable* $|J, p_n|$, *and* $\{a_n P_{n-1} / p_n\} \in BV$, *and if* $\{p_n\}$ *satisfies the same conditions as in Theorem* 7, *then* Σa_n *is absolutely convergent.*

In Chapter V, we investigate into the application of the method $|J, p_n|$ to Fourier series. We prove a couple of theorems, the first of which gives a partial generalization of a theorem of IZUMI and IZUMI [1] and yields a criterion for absolute Abel summability for Fourier series, while the second gives a complete generalization of the same theorem of IZUMI and IZUMI and contains a theorem of MOHANTY and PATNAIK [2] as a special case when $p_n = 1/n$ $(n=1,2,\dots)$.

Writing

$$g^*(t) = \int_t^\pi \frac{f(u)}{\sin u/2}\, du ,$$

our results read :

1) Izumi, M. and Izumi, S. : Proc.Japan Acad., 46(1970), 656-659.

2) Mohanty, R. and Patnaik, J.N. : Jour.London Math.Soc., 43(1968), 452-456.

<u>Theorem 10</u>. <u>Suppose that</u> (i) <u>the sequence</u> $\{p_n\}$ <u>is of bounded variation and that</u> (ii) <u>there exists an</u> a, $1 < a < \rho$, <u>such that</u>

$$(1-x)^{a}\, p(x)\downarrow \text{ as } x \uparrow 1 .$$

<u>If</u> $g^{*}(t)/t^{\rho} p(1-t) \in L\,(0, \pi)$, <u>then the Fourier series of</u> f <u>is</u> $|J, p_n|$<u>-summable at the origin</u> .

<u>Theorem 11</u>. <u>Suppose that</u> (i) <u>the sequence</u> $\{p_n\}$ <u>is of bounded variation</u>, (ii) <u>there exists an</u> a, $0 < a < 1$, <u>such that</u>

$$(1-x)^{a}\, p(x)\downarrow \text{ as } x \uparrow 1 ,$$

<u>and for</u> $z = xe^{it}$,

$$\text{(iii)} \quad p'(z) = O(\,p(z)\,) , \text{ as } z \to 1$$

and

$$\text{(iv)} \quad (1-z)\, p'(z) = O(p(z)), \text{ as } z \to 1 .$$

<u>If</u> $g^{*}(t)/t p(1-t) \in L(0, \pi)$, <u>then the Fourier series of</u> f <u>is</u> $|J, p_n|$<u>-summable at the origin</u>.

We note that we get the following result as a special case of Theorem 10, when $p_n = A_n^{k-1}$, $-1 < k \leq 0$, and, in particular, it gives a criterion for absolute Abel summability.

Theorem 12. Let $-1 < k \leq 0$. *Suppose that the sequence* $\{p_n\}$ *is of bounded variation. If* $g^*(t)/t^{1-k} \in L(0,\pi)$, *then the Fourier series of* f *is* $|A_k|$*-summable at the origin.*

In Chapter VI, we further generalize our results of Chapter V for $|J, p_n|_k$-summability of Fourier series for $k \geq 1$. The results proved in this chapter are :

Theorem 13. Let $k \geq 1$. *Suppose that* (i) *the sequence* $\{p_n\}$ *is of bounded variation,* (ii) *there is an* a, $1<a/k<2$, *such that*

$$(1-x)^{a/k}\, p(x) \downarrow \text{ as } x \uparrow 1,$$

and that (iii) *for* $k > 1$, *the function* $\{(1-x)p(x)\}^{1-k}$ *is bounded in* $(c,1)$.[1] *If* $g^*(t)/t^2 p(1-t) \in L(0, \pi)$, *then the Fourier series of* f *is* $|J, p_n|_k$*-summable at the origin.*

Theorem 14. Let $k \geq 1$. *Suppose that* (i) *the sequence* $\{p_n\}$ *is of bounded variation,* (ii) *there is an* a, $0<a/k<1$, *such that*

$$(1-x)^{a/k}\, p(x) \downarrow \text{ as } x \uparrow 1\,,$$

and for $z = xe^{it}$

$$\text{(iii) } p'(z) = O(p(z)), \text{ as } z \to 1,$$

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1) For $k = 1$, this condition is void.

and

$$\text{(iv)} \quad (1-z)\, p''(z) = O(p(z)), \quad \textit{as} \quad z \to 1.$$

If $g^*(t)/tp(1-t) \in L(0,\pi)$, *then the Fourier series of* f *is* $|J, p_n|_k$*-summable at the origin.*

We deduce in this chapter two interesting results respectively from Theorems 13 and 14 in the special cases when $p_n = 1$, for $n=0,1,2,\ldots$ and $p_n = \frac{1}{n}$ for $n=1,2,\ldots$.

Corollary III. Let $k \geq 1$ *and* $-1 < \alpha \leq 0$. *Suppose that the sequence* $\{p_n\}$ *is of bounded variation. If* $g^*(t)/t^{1-\alpha} \in L(0, \pi)$, *then the Fourier series of* f *is* $|A_\alpha|_k$ *-summable at the origin.*

Corollary IV. Let $k \geq 1$. *Suppose that the sequence* $\{p_n\}$ *is of bounded variation. If*

$$\frac{g^*(t)}{t \log(2\pi/t)} \in L(0,\pi),$$

then the Fourier series of f *is* $|L|_k$*-summable at the origin.*

Chapter VII consists of a theorem for $|J, p_n|_k$ -summability of Fourier series, which includes, as particular case when $k = 1$, one of the other results of IZUMI and IZUMI.[1)]

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1) Izumi, M. and Izumi, S. : Proc.Japan Acad., Theorem 2 , 46(1970), 647-651.

<u>Theorem 15. Suppose that</u> (i) $\{n\, p_n\}$ <u>and</u> $\{n^2\, p_n\}$ <u>are monotone and concave or convex and that</u>

(ii) $(1-x)^{3-1/k}\, p''(x)/p(x) \in L^k(0,\, 1)$. <u>If</u>

$$\int_0^1 \frac{G^k(t)}{t^{2k+1}}\, dt \int_{1-t}^1 (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^k dx < \infty ,$$

<u>where</u> $G(t) = \left(\int_0^t |g(u)|^k du\right)^{1/k}$, $g(t) = \int_t^\pi \frac{\phi(u)}{\sin u/2}\, du$, <u>and</u>

$\phi(u) = \frac{1}{2}\left\{f(x_0+u)+f(x_0-u)-2s\right\}$, <u>then the Fourier series of</u> f <u>is</u> $|J,\, p_n|_k$-<u>summable at the point</u> x_0 .

Recently, PATI and RAMANUJAN[1)] demonstrated the truth of a number of inclusion relations of the type : $|A| \subseteq |AB|$, for different pairs (A, B), where AB is the iteration product of the summation methods A and B, either the methods A and B being based on function-to-function transformations, or A, generally, being a method based on power series, and B a sequence-to-sequence method.

In the last chapter, the author establishes this inclusion relation, when A is the logarithmic method (L) and B, the regular sequence-to-sequence quasi-Hausdorff method

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1) Pati, T. and Ramanujan, M.S. : Bull.U.M.I.,(3), 17(1962), 385-393.

(R, ν) for a bounded sequence under certain restriction. The theorems proved are analogues of a result of ISHIGURO [1] for the corresponding ordinary inclusion $(L) \subseteq (L, (H^*, \chi))$.

<u>Theorem 16.</u> <u>Let</u> (H^*, χ) <u>be a regular quasi-Hausdorff method.</u> <u>If the sequence</u> $\{s_n\}$ <u>is bounded,</u> <u>and if</u>

$$\int_0^1 |\log t| \, |d\chi(t)|$$

is finite, then $|L| \subseteq |L, (H^*, \chi)|$.

<u>Theorem 17.</u> <u>Let</u> (H^*, χ) <u>be a regular quasi-Hausdorff method.</u> <u>If the sequence</u> $\{s_n\}$ <u>is bounded,</u> <u>and if</u>

$$\int_0^{\sigma} |\log t| \, |d\chi(t)|$$

is finite for a positive σ (<1), then $|L| \subseteq |L, (H^*, \chi)|$.

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1) Ishiguro, K. : Proc. Japan Acad., 38 (1962) [illegible].

PREFACE

The present Thesis, [illegible] BASED ON [illegible], is the outcome of my researches that I have been pursuing since 197[illegible], under the esteemed guidance of Dr. [illegible] Ahmed, M.Sc., M.Phil., D.Sc., Reader, Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh.

It has been my proud privilege to have accomplished my researches under the efficient supervision of Dr. [illegible] Ahmad and I take this opportunity to acknowledge my deep sense of gratitude and indebtedness to him for his valuable suggestions, inspiring guidance and constant encouragement throughout the course of these researches.

The Thesis consists of eight chapters. In the first chapter, besides the definitions and notations of the absolute summability methods that are involved in our subsequent discussions, we give a brief résumé of more important results which have their relevance in the context of the subject matter of our investigations. Chapter II is devoted to the study of some properties of $|L|$ and $|A, \lambda|$-methods and their inter-relations. Chapter III deals with the b-translativity of the method (J, p_n). Chapter IV is concerned with the study of some Tauberian theorems for $|J, p_n|$-summability. In Chapter V, we obtain a couple of results on the absolute summability of Fourier series by $|J, p_n|$-methods, one of which yields a criterion for $|A|$-summability. In Chapter VI, we generalise the results of Chapter V for summability $|J, p_n|_k$, $k \geq 1$, of Fourier series. Chapter VII consists of a theorem for summability $|J, p_n|_k$ of Fourier series. The last Chapter centres around the study of a problem on the product of absolute summability methods. Towards the end, a comprehensive bibliography of various publications to which reference has been made in the body of the thesis, is given.

The result of Chapter VII has already been accepted for publication and I attach herewith the attested copy of the letter of acceptance in the Appendix. The materials of other chapters in the form of papers have also been communicated for publication in various international mathematical journals and I propose to present some of them at the ensuing session of the Indian Science Congress Association at Delhi in January, 1976 .

I owe a great deal to Professor [illegible].I. Husain, Head of the Department of Mathematics and Statistics, Aligarh Muslim University, and take this occasion to express profound thanks to him for his constant encouragement

M.A. Rahiman

Mohamed Abdul Rahiman

CONTENTS

CONVENTIONS AND NOTATIONS

SUMMATION CONVENTIONS:

Σ, written without limits, usually denotes $\sum_{0}^{\infty}$, or $\sum_{1}^{\infty}$ if a term of zero is not defined.

$\sum_{n} a_n$ is the sum of all a_n's which are defined.

BINOMIAL COEFFICIENTS:

For $n = 0, 1, 2, ---$, A_n^{α} is defined by the identity:

$$\sum A_n^{\alpha} x^n = (1-x)^{-\alpha-1} \quad (|x| < 1),$$

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2) \ldots\ldots (\alpha+n)}{n!} \quad (\alpha > -1),$$

$$A_n^{-\alpha} = 0 \qquad (n \geq \alpha, \quad \alpha = 1, 2, \ldots\ldots).$$

CONSTANT:

K denotes an absolute constant independent of the variable under consideration, but is not necessarily the same at each occurrence.

O, o and $\sim$:

If $g > 0$, then

$f = O(g)$ means $|f| < Kg$,

$f = o(g)$ means $f/g \to 0$.

In particular,

$f = O(1)$ means that f is bounded and

$f = o(1)$ means that $f \to 0$.

If P and Q are two equivalent summability processes, then we write

$$P \sim Q .$$

Similarly we interpret

$$|P| \sim |Q| .$$

INTEGRAL PART OF x :

$[x]$ denotes the algebraically greatest integer not exceeding x

FINITE DIFFERENCES:

For any sequence $\{s_n\}$,

$$\Delta s_n = s_n - s_{n+1}, \quad \Delta^{0} s_n = s_n ;$$

and

$$\bar{\Delta} s_n = s_n - s_{n-1}, \quad \bar{\Delta}^{0} s_n = s_n .$$

BOUNDED VARIATION:

By ' $\{f_n\} \in BV$ ', we mean that the sequence $\{f_n\}$ is of bounded variation, that is to say,

$$\sum |f_n - f_{n-1}| \leq K,$$

that is,

$$\sum |\bar{\Delta} f_n| \leq K .$$

By ' $f(x) \in BV(h,k)$ ', we mean that $f(x)$ is a function of bounded variation in the interval (h, k).

THE CLASSES L AND L^k $(k > 0)$:

By '$f(x) \in L^k(a,b)$', we mean that the function $f(x)$ belongs to the class of functions which is such that

$|f(x)|^k$, when $k > 0$, is integrable in the sense of Lebesgue over (a, b). We write '$f(x) \in L(a,b)$' for '$f(x) \in L^1(a,b)$.'

Apart from these, all notations and conventions of Chapter I will be adhered to throughout the rest of the Thesis without specific mention, unless otherwise stated.

CHAPTER I

INTRODUCTION

1.1 It was the pioneering work of CAUCHY [1] and the genius researches of ABEL [2] that the foundations of a rigorous theory of infinite series were laid. Although what is now known as the classical principle of Cauchy convergence clearly divided infinite series into two classes, viz., those which have a finite (and unique) sum in the sense of Cauchy and those that fail to have, there remained to be precisely apprehended the distinction between properly divergent series and series with finitely oscillatory partial sums. Towards the end of the nineteenth century, a large variety of oscillatory series were brought within the framework of a sound mathematical interpretation through the concept of summability.

Summability is a generalization of the notion of Cauchy convergence [3] in the sense that the "partial sums" is to

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1) Cauchy (1).

2) Abel (1).

3) Hobson (1), page 65.

be replaced by a suitable average of it in a certain prescribed manner . For the pioneering studies that led to the formulation of the theory of summability , credit goes 'inter alia' to [illegible] , [illegible] , [illegible] , HAUSDORFF [illegible] and others. [1]

Analogously, there emerged the concept of Absolute summability as a natural generalisation of the notion of absolute convergence. Since 1911 , when absolute Cesàro summability was first introduced by [illegible], [2] many contributions have been made by various workers to the theory and application of " Absolute Summability."

The present thesis consists of the recent investigations of the author into the theory and applications of Absolute summability of infinite series by methods based on power series .

1.2 Let $\sum a_n$ be a given infinite series , with sequence $\{s_n\}$ of partial sums .

Generally, all commonly used processes of summability

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1) cf. Hardy (1) ; see also Ishiguro (3) , (4) .

2) The earliest work known to us is that of Fekete (1)

belong to either one or the other of two kinds of processes, viz., the T-processes and the Ø-processes. A T-process is based upon the formation of a sequence of auxiliary means defined by the sequence-to-sequence transformation :

$$(1.2.1) \qquad t_n = \sum_{k=0}^{n} a_{nk} s_k \qquad (n=0, 1, ---),$$

a_{nk} being the element of the nth row and k th column of the Toeplitz matrix $A = (a_{nk})$ of the transformation (1.2.1) .

Other types of transformations under this category are the series-to-sequence transformation , the sequence-to-series transformation and the series-to-series transformation , with which we are not concerned here .

A Ø-process is based upon the formation of the functional transformation defined by sequence-to-function transformation:

$$(1.2.2) \qquad \Phi(x) = \sum_{n=0}^{\infty} \phi_n(x)\, s_n ,$$

or , by series-to-function transformation :

$$(1.2.3) \qquad \Phi(x) = \sum_{n=0}^{\infty} \bar{\phi}_n(x)\, a_n , \quad \bar{\phi}_n(x) = \sum_{k=0}^{n} \phi_n(x) ,$$

where x is a continuous parameter and $\phi_n(x)$ or $\bar{\phi}_n(x)$ is defined over an appropriate interval of x . Similarly , we have one more transformation under this category , viz., function-to-function transformation , with which we are not concerned here .

The series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be summable to a finite number s by a T-process or a ϕ-process according as the sequence $\{t_n\}$, or the function $\Phi(x)$, tends to s as n tends to infinity , or as x tends to the appropriate limit , depending upon the method . 1)

The series is said to be absolutely convergent if $|a_n| = \Sigma\, |s_n - s_{n-1}| < \infty$, that is , if $\{s_n\} \in BV$. Of course, absolute convergence implies convergence.

In analogy with the concept of absolute convergence, the series $\Sigma\, a_n$, or the sequence $\{s_n\}$, is said to be absolutely summable by a T-process if $\{t_n\} \in BV$; and if, in addition, $t_n \to s$, as $n \to \infty$, it is said to be summable $|T|$, to the finite sum s .

Absolute summability by $\emptyset$-process is similarly defined, with the obvious difference that , in this case , $\Phi(x) \in BV(A,a)$,

1) Knopp (1) , page 474 .

where (A, ∞) is a suitable interval of variation of the continuous variable x .

1.3 Given two processes of summability (or absolute summability) , P and Q , Q is said to include P , or P to be included in Q , if every sequence summable by P is also summable by Q, and we write using set-theoretic notations, $P \subseteq Q$, or $Q \supseteq P$.

If $P \subseteq Q$ and $Q \subseteq P$, the two processes P and Q are said to be equivalent and we write $P \sim Q$. If $P \subseteq Q$ and there exists a sequence which is summable Q but not summable P , then we write $P \subset Q$.

A method P is said to be b-translative if , for any bounded sequence $\{s_n\}$, $\{s_n\}$ is summable P if, and only if , $\{s_{n-1}\}$ is summable P ; and it is said to be translative, if the restriction of boundedness on $\{s_n\}$ is dropped.

If P is any T-process or a Φ-process and Q a T-process or P a Φ-process and Q a Φ-process, then the P can be applied to the Q-transform of the sequence under consideration. This iteration-product of these two methods is denoted by PQ , and is called the 'product' of P and Q in this order . The product is evidently non-commutative in general .

A method of summability P is said to be conservative, briefly, P is K, if $(C, 0) \subseteq P$, i.e., the convergence of any series implies its summability P. A method P is said to be regular, briefly P is T, if P is K and also preserves sums of convergent series. A method of summability P is said to be absolutely conservative, briefly, P is AK, if $|C, 0| \subseteq |P|$, i.e., the absolute convergence of any infinite series implies its summability $|P|$, and is said to be absolutely regular, briefly P is AT, if (i) P is AK and (ii) P is T. It has been observed by Miss MORLEY [1] that a method may be AK without being K. We mention in passing that regular matrix methods cannot take care of even all bounded sequences, since, as proved by STEINHAUS, [2] given any regular matrix method M, there exists a bounded sequence which is not summable M. By analogy, one might be led to ask whether it is true that no absolutely regular matrix method can sum all conditionally convergent series.[3]

Necessary and sufficient conditions that a matrix method be AK were first obtained by Miss MEARS [4] in 1937, and functional analytic proofs of equivalent results were given

1) Morley (1).

2) See Cooke (1).

3) Pati (3).

4) Mears (1).

later on by KNOPP and LORENTZ [1] and SUNOUCHI [2] in 1949.[3] Similarly, we have necessary and sufficient conditions that a ϕ-process is K [4] and A.[5]

Now, in the case in which $P \subseteq Q$, but $Q \subseteq P$ is false, that is, $P \subset Q$, the following question can be raised : would it be possible in some manner to restrict the order of magnitude of the terms of the series Σa_n so that, for it $Q \subseteq P$ (and in effect $P \sim Q$)? The results answering this question in the affirmative are called 'Tauberian' . A result of the type $P \subseteq Q$ or $P \subset Q$ is called 'Abelian' .

The aim of the present Thesis is to investigate the problems concerning the results of the types mentioned above (i.e., Abelian theorems, Tauberian theorems, the theorems on product of summability methods) with special reference to certain absolute ϕ-methods and also to consider application of these methods to Fourier series .

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1) Knopp and Lorentz (1) .

2) Sunouchi (1) .

3) These are perhaps the first applications of Functional Analysis to Absolute Summability.

4) Hardy (1) .

5) Ahmad (1) , (4) ; Das (1) .

In the sequel, presenting the definitions and notations of the absolute summability methods that are involved in the present work, the author proposes to give a brief résumé of the hitherto obtained results against the background of which the problems studied in the present thesis suggest themselves.

1.4 Special Absolute Summability Methods.

Special |Z|-methods.

In the special cases, the transform t_n of (1.2.1) reduces respectively to

(a) (C, α)-transform, s_n^{α}, [1] $\alpha > -1$: when

$$a_{nk} = \begin{cases} \binom{n-k+\alpha-1}{n-k} \Big/ \binom{n+\alpha}{n}, & k \le n, \\ 0, & k > n; \end{cases}$$

(b) $(\bar{N}, p_n)$-transform, σ_n : [2] when

$$a_{nk} = \begin{cases} p_k / P_n, & k \le n, \\ 0, & k > n, \end{cases}$$

1) Hardy (1), page 96.

2) Hardy (1), page 57.

where $\{p_n\}$ is a sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty, \quad \text{as } n \to \infty.$$

$\sigma_n = s_n^1$, whenever $p_n = 1$, for each $n = 0, 1, 2 \ldots$.

(c) (R^*, λ_n, α) [or Discrete Riesz mean] -transform, $R_\lambda^\alpha(\lambda_n)$: when

$$a_{nk} = \begin{cases} \left(1 - \dfrac{\lambda_k}{\lambda_{n+1}}\right)^\alpha, & k \leq n, \quad \alpha \geq 0, \\ 0, & k > n, \end{cases}$$

where $\{\lambda_n\}$ is an increasing sequence tending to ∞ with n.

Thus $(R^*, P_{n-1}, 1)$-transform is the same as $(\bar{N}, p_n)$-transform.

(d) (H^*, χ) [or quasi-Hausdorff] -transform, h_n^* : [1)] when

$$a_{nk} = \begin{cases} \dbinom{k}{n} \displaystyle\int_0^t t^{n+1}(1-t)^{k-n}\, d\chi(t), & k \geq n, \\ 0, & \text{otherwise}, \end{cases}$$

1) Hardy (1).

where $\chi(t) \in BV[0, 1]$.

This transform is regular if

$$\int_{+0}^{1} d\chi(t) = \chi(1) - \chi(+0) = 1 .$$ [1)]

The absolute summability methods associated with the above transforms are respectively, the absolute Cesàro method of order α, absolute weighted arithmetic mean method, absolute discrete Riesz method and absolute quasi-Hausdorff method, and are denoted by :

$|C, \alpha|$, $|\bar{N}, p_n|$, $|R^*, \lambda_n, \alpha|$ and $|H^*, \chi|$ respectively

Evidently, summability $(C, 0)$ is the convergence and summability methods $|C, 0|$ and $|R^*, \lambda_n, 0|$ are the same as absolute convergence. Also by definition $|C, \alpha| \subseteq (C, \alpha)$, for $\alpha > -1$.

<u>Special $|\emptyset|$-methods.</u>

In the special case in which : $\emptyset_n(x) = \dfrac{p_n x^n}{\Sigma p_n x^n}$,

$0 \leq x < 1$, the transform $\Phi(x)$ of (1.2.2) reduces to the (J, p_n)-transform, $J(x)$,[2)] defined by

(1.4.1) $$J(x) = (\Sigma p_n x^n)^{-1} \Sigma p_n s_n x^n ;$$

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1) Hardy (1), pages 247-260.

2) Hardy (1), pages 79-80 .

and the associated method is known as $|J, p_n|$-method.[1] The necessary and sufficient condition for the regularity of (J, p_n)-transform is $\Sigma p_n = \infty$.[2]

We generalise this method to define $|J, p_n|_k$-method.

The series Σa_n, or the sequence $\{s_n\}$, is said to be summable $|J, p_n|_k$, $k \geq 1$, if the series on the right of (1.4.1) converges, and, for $0 < c < 1$,

$$\int_c^1 (1-x)^{k-1} \; |J'(x)|^k \, dx < \infty .$$

$|J, p_n|_1$ is the same as $|J, p_n|$, and for $k > 1$, $|J, p_n|$ and $|J, p_n|_k$ are mutually independent.[3]

In the special cases in which $p_n = A_n^{\alpha}$ $(\alpha > -1)$,[4] and $p_n = \frac{1}{n+1}$, $n = 0, 1, 2, \ldots$, $|J, p_n|$ and $|J, p_n|_k$ reduce respectively to $|A_\alpha|$, $|L|$ and $|A_\alpha|_k$, $|L|_k$. The methods $|A_0|$ and $|A_0|_k$[5] are absolute Abel and generalized absolute Abel methods.

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1) Hardy (1), page 79.

2) Borwein (1), Hardy (1), pages 79-81.

3) Mazhar (1).

4) Borwein (1).

5) Flett (1).

The series Σa_n, or the sequence $\{s_n\}$, is said to be summable $|A, \lambda|$, $\lambda > -1$, if the series

$$g_\lambda(x) = (1-x) \sum_{n=0}^{\infty} s_n^\lambda x^n$$

converges for $0 \leq x < 1$, and $g(x) \in BV\,[\delta, \infty)$, $0 < \delta < 1$. The method $|A.(C, \lambda)|$ is the same as $|A, \lambda|$.

The series Σa_n is said to be absolutely summable by Riesz means of "type λ_n" and "order α" $(\alpha \geq 0)$, or summable $|R, \lambda, \alpha|$, if $R_\lambda^\alpha(x) \in BV\,(h, \infty)$, for some positive constant h, where

$$R_\lambda^\alpha(x) = A_\lambda^\alpha(x) \,/\, x^\alpha ,$$

$$A_\lambda^\alpha(x) = \sum_{\lambda_n < x} (x-\lambda_n)^\alpha a_n = \alpha \int_0^x (x-t)^{\alpha-1} A_\lambda(t)\,dt,$$

$$A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n < x} a_n ,$$

and $\{\lambda_n\}$ is a positive strictly monotonic increasing sequence, diverging to infinity.

The earliest definition of any special method of absolute summability was that of $|C, \alpha|$-summability. Although it was introduced by FEKETE[1] for non-negative integral order, it

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1) *Fekete* (1), (2).

was studied by KOGBETLIANTZ [1] in considerable details, who proved that (i) $|C, \alpha| \subset |C, \beta|$ for every $\beta > \alpha > -1$, while (ii) in general $|C, \alpha| \not\subset |C, \beta|$ if $\beta < \alpha$ and $(C, \alpha) \not\subset |C, \beta|$ for $\beta > \alpha$. The result (i) has been obtained by a shorter method by MORLEY [2] and is also a particular case of a theorem of OBRECHKOFF,[3] which states: $|R, \lambda_n, \alpha| \subseteq |R, \lambda_n, \beta|$ for every $\beta > \alpha \geq 0$, in view of the result of HYSLOP [4] that $|R, n, \alpha| \sim |C, \alpha|$, for $\alpha > 0$. This theorem of OBRECHKOFF is usually called the 'first theorem of consistency' as distinguished from 'second theorem of consistency' for absolute Riesz summability which purports to the assertion $|R, \lambda_n, \alpha| \subseteq |R, \mu_n, \alpha|$, $\alpha > 0$, when λ_n and μ_n are related in a prescribed manner. It is known that [5] $|R, \lambda_n, 1| \sim |\bar{R}, \lambda_n, 1|$. Nothing is definite about the equivalence: $|R, \lambda_n, k| \sim |R^*, \lambda_n, 1|$ as the discrete Riesz method does not display such regular features as the Riesz method, whether ordinary or absolute. [6]

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1) Kogbetliantz (1).

2) Morley (1).

3) Obrechkoff (1).

4) Hyslop (1).

5) Mohanty (1), where he states that it was mentioned to him by Bosanquet. An explicit proof due to Pati is quoted in Iyer (2). Also see Lorentz and Macphail (1).

6) For a detailed discussion, reference may be made to Pati (3), pages 10-11.

We also know that the method $|N^*, p_{n-1}, 1|$ is the same as $|\bar{N}, p_n|$. The method $(\bar{N}, p_n)$ is both T[1] and AT.[2] For regular $(\bar{N}, p_n)$-method, SUNOUCHI[3] proved that : If $p_{n+1}/P_n > q_{n+1}/Q_n$, then $\bar{N}, p_n \quad \bar{N}, q_n$, while PEYERIMHOFF[4] proved an elegant limitation theorem : if $p_{n+1}/P_{n+1} = O(p_n/P_n)$, then $\sum \frac{p_n}{P_n}|a_n| < \infty$.

Now we come to the discussion of $|\phi|$-method. Like $|C, \alpha|$-method, $|A|$-method is the most fundamental. By definition, it is evident that $|A| \subset (A)$. Analogous to Abel's classical theorem, we also have the result that $|C, 0| \subseteq |A|$.[5] FEKETE[6] generalized this and proved that : $|C, \alpha| \subseteq |A|$, however large α (>0) may be, and also showed by means of a negative example that $|A| \nsubseteq (C, \alpha)$, and hence $|A| \nsubseteq |C, \alpha|$, however large $\alpha (>0)$ may be. This was also independently verified for Fourier series by RANDELS.[7] It has been demonstrated by PATI[8] that, for the conjugate series of a Fourier series, summability $|A|$ at a point, even when combined with everywhere convergence, does not

1) Hardy (1), page 57.

2) Sunouchi (1).

3) Sunouchi (1).

4) Peyerimhoff (1).

5) Whittaker (1).

6) Fekete (3).

7) Randels (1).

8) Pati (1).

necessarily imply summability $|C, 1|$ at that point .

On the discovery of the fact that Dini's convergence criterion for a Fourier series at a point is sufficient to ensure its summability $|A|$, WHITTAKER [1] was led to the consideration of the inter-relation between summability $(C, 0)$, i.e., convergence, and summability $|A|$. Using an example suggested by LITTLEWOOD, he proved that $(C, 0) \not\subseteq |A|$. PRASAD, [2] on the other hand proved that $|A| \not\subseteq (C, 0)$. HYSLOP [3] has proved that : if, for series $\Sigma\, a_n$, $\Sigma\, \Delta\,(n\, a_n)$ is summable $|C, \alpha+1|$, then $|A| \subseteq |C, \alpha|$, for $\alpha \geq 0$; in particular, if $\{n\, a_n\} \in BV$, then $|A| \subseteq |C, 0|$.

Concerning $|\phi|$-method, i.e., the absolute summability methods based on power series, AHMAD [4] has recently studied a number of problems, e.g., he has proved that : (i) $|A| \subset |L|$, (ii) $|A, \lambda| \subseteq |A, \lambda+\delta|$, for $\lambda > -1$, $\delta > 0$, (iii) $|A_\alpha| \subseteq |A_\beta|$, for $\alpha > \beta \geq -1$, (iv) (J, p_n)-method is AT, whenever $\Sigma\, p_n = \infty$, (v) $|\bar{N}, p_n| \subset |J, p_n|$; in particular, $|\ell| \subset |L|$.[5]

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1) Whittaker (1).

2) Prasad (1) .

3) Hyslop (2) .

4) Ahmad (1), (3), (4).

5) 'logarithmic mean transform', ℓ_n, is defined by :

$$\ell_0 = s_0 \ , \ \ell_1 = s_1$$

$$\ell_n = (\log n)^{-1}\left(s_0 + \frac{s_1}{2} + \ldots + \frac{s_n}{n+1}\right), \ n = 2, 3, \ldots .$$

In Chapter II of the present Thesis, the author studies some properties of $|L|$ and $|A, \lambda|$-methods and proper inclusion : $|A, \lambda| \subset |A, \lambda+\delta|$, for $\lambda > -1$, $\delta > 0$. The result (i) of A.k.A. has also been generalized . While in Chapter IV, he proves three Tauberian theorems for $|J, p_n|$-summability, one of which includes as a special case a particular case of HYSLOP's result mentioned above. The b-translativity of the method (J, p_n) has been studied in Chapter III .

1.5 Application to Fourier Series.

Let $f(t)$ be a periodic function with period 2π , and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Then the Fourier series of $f(t)$ is given by :

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t) \ ,$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt \ ,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt \ , \quad n = 1, 2, \ldots\ldots \ ,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt \ , \quad n = 1, 2, \ldots\ldots \ .$$

Hence, the Fourier series of $f(t)$, at $t=x$, is $\Sigma A_n(x_0)$, and will be denoted by $\mathfrak{S}[f]_{x_0}$. Let us write

$$\tilde{\phi}(t) = \tilde{\phi}_{x_0}(t) = \frac{1}{2}\left\{f(x_0+t) + f(x_0-t)\right\} .$$

It is easily seen that $\mathfrak{S}[f]_{x_0}$ is the same as $\mathfrak{S}[\tilde{\phi}]_0$. If, in addition, the function f is even, then $\tilde{\phi}(t)=f(t)$.

We also write

$$\tilde{\phi}_1(t) = t^{-1}\,\Phi(t) = \frac{1}{t}\int_0^t \tilde{\phi}(u)\,du ,$$

$$\phi_{x_0}(t) = \frac{1}{2}\left\{f(x_0+t) + f(x_0-t) - 2s\right\}.$$

whenever $s=0$, $\phi_{x_0}(t) = \tilde{\phi}_{x_0}(t)$.

We have already referred to the result of Whittaker that Dini's convergence criterion for Fourier series at a point is sufficient for its summability $|A|$, i.e., if, for some $\eta > 0$, $\int_0^\eta t^{-1}|\phi(t)|\,dt$ exists, then $\mathfrak{S}[f]_{x_0}$ is summable $|A|$ to s. PRASAD[1)] showed that Jordan's convergence criterion is also a criterion for the summability $|A|$ of a Fourier series at a point, that is, if

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1) Prasad(1) .

$\not{}(t) \in BV(0, \eta)$, for some $\eta > 0$, then $S[f]_{x_0}$ is summable $|A|$. [redacted][1] improved this result by showing that Jordan's condition may be replaced by de la Vallée-Poussin's condition : $\not{}_1(t) \in BV(0, \eta)$, $\eta > 0$. [redacted][2] also established results on $|A|$-summability of Fourier series under one of the conditions : (i) $\not{}(t)$ is absolutely continuous in $(0, \eta)$, for some $\eta > 0$, (ii) $\int_0^{\eta} |\Phi(t)| t^{-2} dt$ exists for $\eta > 0$. The condition (ii) includes [redacted]'s condition, while (i) is independent of other conditions .

On the other hand, for $|L|$-summability, [redacted] and [redacted][3] proved that : if

$$\frac{1}{t \log(2\pi/t)} \int_t^{\pi} \frac{f(u)}{2\sin u/2}\, du = \frac{\overset{*}{g}(t)}{t \log(2\pi/t)} \in L\,(0, \pi)\ ,$$

the $S[f]_{x_0}$ is summable $|L|$. Recently, IZUMI and IZUMI[4] have generalized this result for $|J, p_n|$ in a couple of directions. In Chapter V, we further generalize their theorem establishing a couple of theorems, one of which yields as a special case, a criterion for $|A|$-summability of Fourier series.

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1) Mishra (1) .

2) Prasad (1) .

3) Mohanty and Patnaik (1) .

4) Izumi and Izumi (2) .

*) These results were all generalized by Bosanquet [Proc. Edinburgh Math. Soc.(2), 4(1934), 12-17] who proved that, if for $\alpha \geq 0$, $\eta > 0$, $\not{}_{\alpha}(t) \in BV(0, \eta)$, then $S[f]_{x_0}$ is summable $|A|$, where $\not{}_{\alpha}(t) = \frac{\alpha}{t^{\alpha}}\int_0^t (t-u)^{\alpha-1} \not{}(u)\,du$, $\alpha > 0$, and $\not{}_0(t) = \not{}(t)$.

In Chapter VI, we further generalize our results of Chapter V for $|J, p_n|_k$-method, $k \geq 1$, while, in the last but one chapter, we establish a theorem on $|J, p_n|_k$-summability of Fourier series, which includes, as a particular case when $k=1$, one of the other results of Izumi and Izumi[1].

1.6 Product of Summability Methods.

Let A and B be two summability methods for sequences $\{s_n\}$, and let us denote by AB the iteration-product which associates with any given sequence the A-transform of its B-transform (of course, provided it is possible to define it).

In 1952, the following question was raised by Szász: if a sequence $\{s_n\}$ is summable by the A method, then, is the B-transform of $\{s_n\}$, where B is a regular sequence-to-sequence method, also summable by the A method to the same sum as before? Since then, Szász and several other authors have answered this question in the affirmative for various pairs (A, B)[2] of summability methods, e.g., Szász[3] for

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1) Izumi and Izumi (1) .

2) By (A, B), we mean an ordered pair of summability methods A and B , for which $A \subseteq A.B$.

3) Szász(1) , (2) .

(Abel, Hausdorff), (Borel, Hausdorff), (Laplace, Riesz), (Abel, circle method), (Abel, S_q-method); RAMANUJAN[1] for (Abel, quasi-Hausdorff), (Borel, quasi-Hausdorff), (Abel, (J^*, ψ)-method); PARAMESWARAN[2] for (Nörlund, Nörlund type); BORWEIN[3] for (Logarithmic, Hausdorff), (A_K, Hausdorff); ISHIGURO[4] for (Logarithmic, quasi-Hausdorff), (Logarithmic, (J^*, ψ)-method), and AHMAD[5] for((J, p_n), quasi-Hausdorff). RAJAGOPAL,[6] PATI[7] and PATI and LAL[8] also proved several theorems.

Similarly, there arises an analogous question for absolute summability, i.e., the question of determining under what circumstances $|A| \subseteq |B|$? This question has recently been answered in the affirmative by PATI and RAMANUJAN[9] for various pairs(A, B) .

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1) Ramanujan (2), (5).
2) Parameswaran (1).
3) Borwein (2), (3).
4) Ishiguro (1), (2).
5) Ahmad (1), Chapter IX.
6) Rajagopal (1).
7) Pati (2).
8) Pati and Lal (1).
9) Pati and Ramanujan (1).

Concerning the product of summability methods, [illegible] [1] proved the following : Let (H^*, χ) be a regular quasi-Hausdorff method. If $\{s_n\}$ is bounded and $\int_0^{\sigma} \log t \, |d\chi(t)|$ is finite for a positive $\sigma (\leq 1)$, then $L \subseteq L.(H^*, \chi)$.

In the last chapter of the present Thesis, we obtain an analogous result for absolute summability

CHAPTER II

ON $|L|$- AND $|C,\lambda|$-SUMMABILITY METHODS

2.1 __Definitions and Notations__. Let $\{s_n\}$ be a sequence of complex numbers such that $s_0 \neq 0$, and let $\{s_n^{\lambda}\}$ be the sequence of associated (C, λ)-mean, i.e.,

$$(2.1.1) \qquad s_n^{\lambda} = \binom{n+\lambda}{n}^{-1} \sum_{v=0}^{n} \binom{v+\lambda-1}{v} s_{n-v} \quad (\lambda > -1).$$

Let us write

$$(2.1.2) \qquad L(x) = \left(\log \frac{1}{1-x}\right)^{-1} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^n , \quad (0 \leq x < 1);$$

$$(2.1.3) \qquad g_{\lambda}(x) = (1-x) \sum_{n=0}^{\infty} s_n^{\lambda} x^n , \quad (\lambda > -1 ; 0 \leq x < 1);$$

and

$$(2.1.4) \qquad G_{\lambda}(y) = g_{\lambda}\left(\frac{y}{1+y}\right) = (1+y)^{-1} \sum_{n=0}^{\infty} s_n^{\lambda} \left(\frac{y}{1+y}\right)^{n}, (\lambda > -1; 0 \leq x < 1).$$

If

$$(2.1.5) \qquad L(x) \; [\text{or } g_\lambda(x)] \to s, \quad \text{as } x \to 1-0,$$

the sequence $\{s_n\}$ is said to be summable to the value s by the Logarithmic method (L) [or by the (A, λ)-method], or simply summable (L) [(A, λ)] to s, and we also write $s_n \to s(L)$ [$s(A, \lambda)$].[1] The summability $(A, 0)$ is the same as Abel summability (A).

If the series on the right of (2.1.2) [or (2.1.3)] converges for $0 \leq x < 1$, and if

$$(2.1.6) \qquad L(x) \in BV(\delta, 1) \;\; [\text{or } g_\lambda(x) \in BV(\delta, 1)], \quad 0 < \delta < 1,$$

we say that $\{s_n\}$ is absolutely summable (L) [(A, λ)], or simply summable $|L|$ [$|A, \lambda|$];[2] and if in addition (2.1.5) also holds, then, we say that it is summable $|L|_s$ [$|A, \lambda|_s$] and also write $s_n \to s|L|$ [$s|A, \lambda|$]. The method $|A, 0|$ is the same as absolute Abel method $|A|$.

The sequence $\{s_n\}$ is also said to be summable $|A, \lambda|$, if the series on the right of (2.1.4) converges for all $y > 0$, and $G_\lambda(y) \in BV(0, \infty)$.

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1) Borwein (3).

2) Ahmad (1), Chapter VIII.

We also use the identity :[1)]

$$(2.1.7) \quad \theta_{\lambda+\delta}(y) = \frac{\Gamma(\lambda+\delta+1)}{\Gamma(\delta)\,\Gamma(\lambda+1)}\, y^{-\lambda-\delta} \int_0^y (y-t)^{\delta-1}\, t^{\lambda}\, \theta_\lambda(t)\, dt \;;$$

for $\lambda > -1$, $\delta > 0$.

2.2 Introduction. Recently, BORWEIN[2)] has investigated some properties of (L)-method, and in particular, considered its relationship to the (A, λ)-method. In the present chapter we propose to investigate the analogous properties of $|L|$ -method and to consider its relation to $|A, \lambda|$-method. It is to be noted that , very recently , AHMAD[3)] has studied the relation of $|L|$-method with certain other absolute summability methods .

2.3 Translativity. In this section , we prove

Theorem 1. The method $|L|$ is translative .

By this we mean $s_{n+1} \to s\,|L|$ if , and only if $s_n \to s\,|L|$.

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1) Borwein (3) , page 214 , identity (1); for $\lambda \geq 0$, this result is due to Kogbetliantz (1) , page 37 ; see also Lord (1) , page 243 .

2) Borwein (3) .

3) Ahmad (4) .

We require the following lemmas for its proof.

Lemma 1.[1] *The method* $(\ )$ *is translative.*

Lemma 2.[2] *Let* $\alpha_0 \neq 0$ *and the power series* $\sum_{n=0}^{\infty} \alpha_n x^n$ *be convergent for* $0 \leq x < \rho$ $(\rho \leq \infty)$. *If*, *uniformly for* $n \geq 0$,

$$\frac{\sum_{\nu=0}^{n} \alpha_\nu x^\nu}{\sum_{\nu=0}^{\infty} \alpha_\nu x^\nu} \in BV_x(0, \rho),$$

then

$$\frac{\sum_{n=0}^{\infty} \beta_n x^n}{\sum_{n=0}^{\infty} \alpha_n x^n} \in BV(0, \rho),$$

whenever $\{\beta_n / \alpha_n\} \in BV$.

Lemma 3.[3] *If the power series* $\sum_{n=0}^{\infty} \alpha_n x^n$ *is the same as in Lemma 2*, *then uniformly in* $n \geq 0$,

$$\alpha_n(x) = \frac{\sum_{\nu=0}^{n} \alpha_\nu x^\nu}{\sum_{\nu=0}^{\infty} \alpha_\nu x^\nu} \in BV_x(0, \rho).$$

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1) Borwein (3), Theorem 1.

2) Ahmad (3), Lemma 1.

3) Ahmad (3), Lemma 2.

Lemma 4. *If the sequence* $\{s_n\}$ $(s_0 \neq 0)$ *is summable* $|L|$, *and the sequence* $\{\varepsilon_n\}$ *is such that* $\sum_{n=0}^{\infty} |\Delta\varepsilon_n| < \infty$, *then* $\{s_n \varepsilon_n\}$ *is summable* $|L|$.

Proof. The lemma immediately follows from the identity :

$$(\log\frac{1}{1-x})^{-1} \sum_{n=0}^{\infty} \frac{s_n\varepsilon_n}{n+1} x^{n+1} = \left\{ \frac{\sum_{n=0}^{\infty} \frac{s_n\varepsilon_n}{n+1} x^{n+1}}{\sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}} \right\} \left\{ \frac{\sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}}{\log(\frac{1}{1-x})} \right\},$$

by virtue of the hypothesis that $\{\varepsilon_n\} \subset BV$ and by an appeal to Lemmas 2 and 3 .

Proof of Theorem 1. Suppose that $s_n \to s\,|L|$, and note that , for $0 \leq x < 1$,

$$(2.3.1) \quad \sum_{n=0}^{\infty} \frac{s_{n+1}}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{s_n}{n+1} x^n + \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)} x^n$$

and

$$(2.3.2) \quad \sum_{n=1}^{\infty} \frac{s_{n-1}}{n+1} x^n = \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} - \sum_{n=0}^{\infty} \frac{s_n}{(n+1)(n+2)} x^{n+1}.$$

Applying Lemma 3 , with $\varepsilon_n = \frac{1}{n}$ or $\frac{1}{n+2}$, we deduce , from the identity (2.3.1) , that $\{s_{n+1}\}$ is summable $|L|$, and, from the identity (2.3.2), that $\{s_{n-1}\}$ is summable $|L|$.

Again , since , from Lemma 1, $s_{n+1} \to s(L)$ if, and only if $s_n \to s(\ )$, we conclude that $s_{n+1} \to s\,|L|$ and $s_{n-1} \to s\,|L|$, from $s_n \to s\,|L|$.

Hence the result.

2.4 Relationship between $|L|$ and $|A, \lambda|$ -methods.

We commence this section with some results on the $|A, \lambda|$ - method. We prove

Theorem 2. For $\lambda > -1$, $\delta > 0$, $|A, \lambda| \subset |A, \lambda+\delta|$.

The inclusion relation $|A, \lambda| \subseteq |A, \lambda+\delta|$ has recently been established by AHMAD.[1] In order to prove our theorem, we combine this result with the following.

Theorem 3. There is an $|A, \lambda+\delta|$ -summable sequence which is not $|A, \lambda|$ -summable.

Proof. Let the sequence $\{s_n\}$ be defined by

$$\frac{\sin(1+y)}{(1+y)^2} = G_{\lambda+\delta}(y) . \qquad (2.4.1)$$

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1) Ahmad (1), Chapter VIII, Theorem 3.

Then, since $\frac{\sin(1+y)}{(1+y)^2} \in BV(0, \infty)$, $\{s_n\}$ is summable $|A, \lambda+\delta|$.

Now, since

$$G_{\lambda+\delta}(y) = (\lambda+\delta)\, y^{-\lambda-\delta} \int_0^y t^{\lambda+\delta-1}\, G_{\lambda+\delta-1}(t)\, dt ,$$

from (2.4.1), we obtain

$$\frac{y^{\lambda+\delta}}{\lambda+\delta}\, \frac{\sin(1+y)}{(1+y)^2} = \int_0^y t^{\lambda+\delta-1}\, G_{\lambda+\delta-1}(t)\, dt$$

so that, differentiating both sides, we get

$$(2.4.2)\quad G_{\lambda+\delta-1}(y) = \frac{1}{\lambda+\delta}\left[\frac{\cos(1+y)}{(1+y)^2}\, y - 2y\frac{\sin(1+y)}{(1+y)^3} + \frac{\sin(1+y)}{(1+y)^2}(\lambda+\delta)\right].$$

Hence, we see that $G_{\lambda+\delta-1}(y) \notin BV(0, \infty)$ and thus $\{s_n\}$ is not summable $|A, \lambda+\delta-1|$ and a fortiori, not summable $|A, \lambda|$, for any $\lambda > -1$.

The next theorem extends the known result : $|A| \subset |L|$ of AHMAD .[1)]

1) Ahmad (4), Theorem 3(a) .

Theorem 4. For $-1 < \lambda \leq 1$, $|A, \lambda| \subset |L|$.

Proof of Theorem 4. Suppose that $\{s_n\}$ is summable $|A, 1|$ and let $t_n = s_n^1$. Then $\{t_n\}$ is summable $|A|$ and consequently, it is also summable $|L|$. Further,

$$s_{n+1} = t_{n+1} + (n+1)\,(t_{n+1} - t_n)\,,$$

so that, for $0 < x < 1$,

$$\left(\log \frac{1}{1-x}\right)^{-1} \sum_{n=0}^{\infty} \frac{s_{n+1}}{n+1}\, x^n = \left(\log \frac{1}{1-x}\right)^{-1} \sum_{n=0}^{\infty} \frac{t_{n+1}}{n+1}\, x^{n+1} +$$

$$+ (1-x)\left(\log \frac{1}{1-x}\right)^{-1} \sum_{n=0}^{\infty} t_n\, x^n -$$

$$- t_0 \left(\log \frac{1}{1-x}\right)^{-1}.$$

Hence $\{s_{n+1}\}$ is summable $|L|$ and then, in view of Theorem 1, $\{s_n\}$ is also summable $|L|$.

We have thus proved that $|A, 1| \subseteq |L|$, and a fortiori $|A, \lambda| \subseteq |L|$, for $-1 < \lambda \leq 1$. The full result is now the consequence of Theorem 2 and the following:

Theorem 5. There is a $|L|$-summable sequence which is not $|A, \lambda|$-summable, for $-1 < \lambda \leq 1$.

We require the following lemma for the proof of this theorem.

<u>Lemma 5.</u>[1] <u>Let, for</u> $x > 0$, $p(u) \in L(0, x)$, <u>and</u> $P(x) = \int_0^x p(u)\, du \neq 0$. <u>Then a necessary and sufficient condition for</u>

$$h(x) = \frac{1}{P(x)} \int_0^x p(u)\, g(u)\, du \in BV\,[0, \infty),$$

<u>whenever</u> $g(u) \in BV\,[0, \infty)$, <u>is</u>

$$|P(u)| \int_u^\infty \left| \frac{p(x)}{\{P(x)\}^2} \right| dx \leq M,$$

<u>for</u> $u > 0$, <u>where</u> M <u>is a positive constant independent of</u> x.

<u>Proof of Theorem</u> 5. Let the sequence $\{a_n\}$ be defined by:

$$(2.4.3) \qquad (1-x)^{\rho} \cos \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad (|x| < 1).$$

Then

$$(2.4.4) \qquad -\rho(1-x)\cos\frac{1}{1-x} - (1-x)\sin\frac{1}{1-x} = (1-x)\sum_{n=0}^{\infty} a_n x^n .$$

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1) Bosanquet (2), page 280, Result III.

As in the proof of a result of Ahmad,[1] from (2.4.3), we see that $\{s_n\}$ is summable $|A|$, but, by (2.4.4) and the following analysis, we find that $\{s_n\}$ is not summable $|C, 1|$ and a fortiori, not summable $|C, \alpha|$, for any α, $-1 < \alpha < 1$.

From the identity (2.4.4), we have

$$s_0(y) = -\frac{2}{1+y}\cos(1+y) - \frac{1}{1+y}\sin(1+y)$$

and

$$\begin{aligned} s_1(y) &= \frac{1}{y}\int_0^y \left[\frac{-2}{(1+t)^2}\cos(1+t) - \frac{1}{1+t}\sin(1+t)\right] dt \\ &= \frac{-2}{y}\int_0^y \frac{1}{(1+t)^2}\cos(1+t)\,dt - \frac{1}{y}\int_0^y \frac{1}{1+t}\sin(1+t)\,dt \\ &= -I_1(y) - I_2(y), \end{aligned}$$

say, where

$$I_1(y) = \frac{2}{y}\int_0^y \frac{1}{(1+t)^2}\cos(1+t)\,dt$$

and

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1) Ahmad (4), page 548, proof of Theorem 3.

$$I_2(y) = \frac{1}{y}\int_0^y \frac{1}{(1+t)} \sin(1+t)\, dt$$

$$= \frac{y+2}{y(y+2)} \int_0^y (1+t)\left\{\frac{1}{(1+t)^2} \sin(1+t)\right\} dt$$

$$= (y+2)\left[\frac{1}{y(2+y)} \int_0^y (1+t)\left\{\frac{1}{(1+t)^2} \sin(1+t)\right\} dt\right].$$

Now, by virtue of Lemma 5,

$$I_1(y) \in BV[0,\infty), \text{ since } \frac{1}{(1+t)^2} \cos(1+t) \in BV[0,\infty)$$

and

$$I_2(y) \notin BV[0,\infty), \text{ since } (y+2) \notin BV[0,\infty),$$

although

$$\frac{1}{y(2+y)} \int_0^y (1+t) \left\{\frac{1}{(1+t)^2} \sin(1+t)\right\} dt \in BV[0,\infty).$$

Hence, $\sigma_1(y) \notin BV[0,\infty)$, that is, $\{s_n\}$ is not summable $|A, 1|$.

This terminates the proof of Theorem 5.

CHAPTER III

REGULARITY OF (J,p_n)-METHOD

3.1 Definitions and notations. Suppose that $p_n \geq 0$ and that the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n ; \quad p(0) = p_0 ,$$

is 1. Given any series $\sum_{n=0}^{\infty} a_n$, with the sequence of partial sums $\{s_n\}$, we shall use the notations :

$$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n, \quad (0 \leq x < 1),$$

and

$$J(x) = J_s(x) = p_s(x) \,/\, p(x) .$$

If $\lim_{x\to 1-} J(x) = s$, we say that $\sum a_n$ is summable (J,p_n) to s.[1]

The necessary and sufficient condition for the regularity of the method (J, p_n) is that $p(x) \to \infty$, as $x \to 1-$, i.e., $\sum p_n = \infty$.

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1) Hardy (1) , page 80 ; see also Borwein (?) .

In the special cases in which, $p_n = \binom{n+k}{n}$, $k > -1$ and $p_n = (n+1)^{-1}$, for all $n = 0, 1, 2, \ldots$, the method reduces respectively to the methods (A_k) and (L). The method (A_0) is the same as Abel's method.

Let $A = (a_{mn})$ be a limitation matrix [1] and, for any bounded sequence $\{s_n\}$, put

$$t_m = \sum_{n=1}^{\infty} a_{mn} s_n , \quad t'_m = \sum_{n=1}^{\infty} a_{mn} s_{n-1} ,$$

where s_0 is taken to be 0. Then A is called b-translative, if

$$\lim_{m \to \infty} (t_m - t'_m) = 0$$

for all bounded sequences $\{s_n\}$.

If for any sequence $\{s_n\}$, $t_n \to s$ if and only if $t'_n \to s$, then the matrix A is said to be translative.

2.2 Introduction. Recently, BORWEIN [2] has proved the translativity of the methods (A_k) and (L). In the present chapter, we prove a theorem for b-translativity of the method (J, p_n).

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1) Petersen (1).

2) Borwein (1), (3).

3.3 We establish the following theorem.

<u>Theorem. The (J, p_n)-method is b-translative if, and only if</u>

$$\sum_{n=0}^{\infty} |\Delta p_n| \leq K \text{ and } \sum_{n=0}^{\infty} p_n = \infty .$$

We get the following known results from our theorem.

<u>Corollary I</u> . <u>The method</u> (A_k) , $-1 < k \leq 1$, <u>is b-translative</u>. <u>In particular, Abel method is b-translative</u>.

<u>Corollary II</u>. <u>The</u> (L)-<u>method is b-translative</u>.

3.4 The following lemma is needed for the proof of our theorem.

<u>Lemma 1</u>.[1)] The matrix $A = (a_{mn})$ limits all bounded sequences to 0 if an only if $\sum_{n=1}^{\infty} |a_{mn}|$ converges for every m and

$$\sum_{n=1}^{\infty} |a_{mn}| \to 0 \quad \text{as} \quad m \to \infty .$$

3.5 <u>Proof of the theorem</u>. Taking $x = 1 - \frac{1}{m}$ in the

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1) Petersen (1), page 20, Corollary 2 of Theorem 1.5.2

definition of (J, p_n)-method, $\lim_{x\to 1-0} J(x) = s$ is the same as

$$\lim_{m\to\infty} J\left(1-\frac{1}{m}\right) = \lim_{m\to\infty} \frac{\sum_{n=0}^{\infty} p_n s_n \left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n} = s$$

Also, let

$$J'\left(1-\frac{1}{m}\right) = \frac{\sum_{n=0}^{\infty} p_n s_{n-1} \left(1-\frac{1}{m}\right)^n}{\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n}.$$

Then the method (J, p_n) is said to be translative if

$$\lim_{m\to\infty} \left[J\left(1-\frac{1}{m}\right) - J'\left(1-\frac{1}{m}\right) \right] = 0. \tag{3.8.1}$$

Now,

$$J\left(1-\frac{1}{m}\right) - J'\left(1-\frac{1}{m}\right)$$

$$= \frac{1}{p\left(1-\frac{1}{m}\right)} \left[\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n s_n - \sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n s_{n-1} \right]$$

$$= \frac{1}{p\left(1-\frac{1}{m}\right)} \left[\sum_{n=0}^{\infty} p_n \left(1-\frac{1}{m}\right)^n s_n - \sum_{n=0}^{\infty} p_{n+1} \left(1-\frac{1}{m}\right)^{n+1} s_n \right]$$

$$= \frac{1}{p\left(1-\frac{1}{m}\right)} \left[\sum_{n=0}^{\infty} (p_n - p_{n+1}) \left(1-\frac{1}{m}\right)^n s_n + \sum_{n=0}^{\infty} p_{n+1} \frac{1}{m} \left(1-\frac{1}{m}\right)^n s_n \right]$$

$$= \frac{1}{p(1-\frac{1}{m})}\left[\sum_{n=0}^{\infty}(p_n - p_{n+1})(1-\frac{1}{m})^n s_n + \sum_{n=0}^{\infty} p_{n+1}\,\frac{1}{m}(1-\frac{1}{m})^n\; s_n\right]$$

$$= \frac{1}{p(1-\frac{1}{m})}\left[\sum_{n=0}^{\infty}\Delta p_n(1-\frac{1}{m})^n s_n + \sum_{n=0}^{\infty} p_{n+1}\,\frac{1}{m}(1-\frac{1}{m})^n\; s_n\right]$$

$$= \sum_{n=0}^{\infty} a_{mn}\, s_n + \frac{1}{mp(1-\frac{1}{m})}\sum_{n=0}^{\infty} p_{n+1}\;(1-\frac{1}{m})^n\; s_n$$

where

$$a_{mn} = \frac{\Delta p_n\;(1-\frac{1}{m})^n}{p(1-\frac{1}{m})} .$$

But, as $m \to \infty$,

$$\frac{1}{mp(1-\frac{1}{m})}\sum_{n=0}^{\infty} p_{n+1}(1-\frac{1}{m})^n\; s_n \to 0 .$$

Therefore, $(J,\ p_n)$-method is b-translative, that is, (3.5.1) holds true, if and only if

(3.5.2) $$\lim_{m\to\infty}\sum_{n=0}^{\infty} a_{mn}\, s_n = 0 .$$

Applying Lemma 1, the necessary and sufficient conditions for (3.5.2) to hold are

(3.5.3) $$\frac{\sum_{n=0}^{\infty} |\Delta p_n|\;(1-\frac{1}{m})^n}{p(1-\frac{1}{m})}$$

converges for every m and

$$(2.5.4) \qquad \frac{\sum_{n=0}^{\infty} \left|\Delta p_n\right| (1-\frac{1}{m})^n}{p(1-\frac{1}{m})} \to 0, \quad \text{as } m \to \infty,$$

and (2.5.4) holds true if and only if

$$\text{(i)} \quad p(1-\frac{1}{m}) \to \infty, \quad \text{as} \quad n \to \infty$$

that is, $P_n \to \infty$, as $n \to \infty$, and

$$\text{(ii)} \quad \sum_{n=0}^{\infty} \left|\Delta p_n\right| (1-\frac{1}{m}) \leq K, \quad \text{as } m \to \infty,$$

that is, $\sum_{n=0}^{\infty} \left|\Delta p_n\right| \leq \cdot$, as this series cannot tend to zero, as $m \to \infty$.

This completes the proof of the theorem.

CHAPTER IV

TAUBERIAN THEOREMS FOR J, p_n -SUMMABILITY

4.1 <u>Definitions and Notations</u>. We suppose throughout that

$$p_n > 0 \ , \ \sum_{n=0}^{\infty} p_n = \infty \ ,$$

and that the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \ , \quad p(0) = p_0 \ ,$$

is 1. Given any series $\sum_{n=0}^{\infty} a_n$, with the sequence of partial sums $\{s_n\}$, we shall use the notations :

(4.1.1) $$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n \ ,$$

and

(4.1.2) $$J(x) \equiv J_s(x) = p_s(x) \ / \ p(x) \ .$$

If the series on the right of (4.1.1) is convergent in the right open interval $(0, 1)$ and if

$$J(x) \in \mathcal{J}(c, 1), \quad (0 < c < 1),$$

we say that the series $\sum_{n=0}^{\infty} a_n$, or the sequence $\{s_n\}$, is absolutely summable (J, p_n), or simply summable $|J, p_n|$.[1) It is known that the (J, p_n)-transform (4.1.2) is both regular [2) and absolutely regular. [3)

In the special case in which $p(x) = (1-x)^{-1}$, $0 \leq x < 1$, $|J, p_n|$-method reduces to the absolute Abel method, $|A|$.

Now, we write

$$P_n = p_0 + p_1 + \dots + p_n \neq 0, \quad n = 0, 1, 2, \dots,$$

$$P_{-1} = p_{-1} = 0,$$

and

(4.1.3) $$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (n \geq 0).$$

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1) Ahmad (1), Chapter VIII; see also Ahmad (4) and Das(1).

2) Hardy (1), page 80; see also Borwein (2).

3) Ahmad (1), (4).

with $p_n > 0$. If $\{\sigma_n\} \in BV$, we say that the series $\sum a_n$, or the sequence $\{s_n\}$ is absolutely summable $(\bar{N}, p_n)$, or simply summable $|\bar{N}, p_n|$.[1] The $(\bar{N}, p_n)$-transform (4.1.3) is also both regular[1] and absolutely regular.[2]

Again, let us write

$$(4.1.4) \qquad t_n = (P_n)^{-1} \sum_{v=1}^{n} P_{v-1}\, a_v \,, \quad t_0 = 0 \,.$$

Then, from (4.1.3) and (4.1.4), we get

$$(4.1.5) \qquad \bar{\Delta}\sigma_n = \sigma_n - \sigma_{n-1} = \frac{p_n}{P_{n-1}}\, t_n \,(n \geq 1),$$

and

$$(4.1.6) \qquad s_n = \sigma_n + t_n \,(n \geq 0) .$$

Thus the summability $|\bar{N}, p_n|$ of $\{s_n\}$ is the same as

$$(4.1.7) \qquad \sum_n \frac{p_n}{P_{n-1}} \left| t_n \right| < \infty ,$$

and the summability $|\bar{N}, p_n|$ of the sequence $\{ P_{n-1}\, a_n / p_n \}$ is the same as

$$\{t_n\} \in BV .$$

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1) Hardy (1), page 57.

2) Sunouchi (1).

Suppose that $\{\lambda_n\}$ be a sequence such that $0 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$, as $n \to \infty$, and let us write

$$A_\lambda(t) = \overset{0}{A_\lambda}(t) \equiv \sum_{\lambda_n \leq t} a_n .$$

And, for $k > 0$, we write

$$\overset{k}{A_\lambda}(t) = \sum_{\lambda_n \leq t} (t - \lambda_n)^k a_n ,$$

with $\overset{k}{A_\lambda}(t) = 0$, for $t < \lambda_1$.

If

$$(4.1.8) \qquad \overset{k}{R_\lambda}(t) \equiv \overset{k}{A_\lambda}(t) \,/\, t^k \in V(h, \infty) ,$$

for some finite positive number h, then we say that the series $\sum a_n$ is absolutely summable by Riesz means[1] of 'type' λ_n and 'order' k ($k \geq 0$), or simply summable $|R, \lambda_n, k|$.[2]

Now, in $\overset{k}{R_\lambda}(t)$, if t be restricted to the sequence $\{\lambda_n\}$ only, then we obtain the 'discrete Riesz means', or (R^*, λ, k)-means :

$$(4.1.9) \qquad \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right)^k a_\nu , \quad k \geq 0 .$$

---

1) Definition of Riesz mean $\overset{k}{R_\lambda}(t)$ is due to Riesz (1).

2) Obrechkoff (1), (2) .

We say that the series $\sum a_n$ is absolutely summable by Riesz's discrete means of 'order' k and 'type' λ_n, or simply summable $\left| R^*, \lambda_n, k \right|$, $k > 0$, if

$$\sum_{v=0}^{n} \left(1 - \frac{\lambda_v}{\lambda_{n+1}}\right)^k a_v \in BV.^{1)}$$

It may be noted that $\left| R, \lambda_n, 0 \right|$ and $\left| R^*, \lambda_n, 0 \right|$ are the same as absolute convergence.

It is known that the summability methods $\left| R, \lambda_n, 1 \right|$ and $\left| R^*, \lambda_n, 1 \right|$ are equivalent. [2)]

We observe that, by definition, $(R^*, P_{n-1}, 1)$-mean is the same as the $(\bar{N}, p_n)$-mean and in effect, $\left| \bar{N}, p_n \right|$-method is equivalent to $\left| R, P_n, 1 \right|$-method.

We use throughout the notation :

$$(4.1.1) \quad \phi(w) = \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v) P_u Q_{v-u}}{\sum_{v=0}^{\infty} e^{-v/w} \sum_{u=0}^{v} (v - u + 1) P_u Q_{v-u}} .$$

Throughout we use C as a strictly positive constant ; possibly different at each occurrence .

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1) see Mohanty (1) .

2) Mohanty (1), where he states that it was mentioned to him by Prof. L.S.Bosanquet. An explicit proof due to Prof. T.Pati is quoted in Iyer (2) . Also, see Lorentz and Macphail (1) .

4.2 <u>Introduction</u>. Recently, Ahmad[1] proved the following Abelian theorem for $|J, p_n|$-summability :

<u>Theorem A.</u> $|\bar{N}, p_n| \subset |J, p_n|$.

Concerning summability $|\bar{N}, p_n|$, we know the following Tauberian theorem .

<u>Theorem B.</u>[2] <u>If</u> (i) $\sum a_n$ <u>is summable</u> $|R, P_{n-1}, 1|$, (ii) $\{t_n\} \in BV$ <u>and</u> (iii) $\{P_{n-1}/P_n\} \subset BV$, <u>then</u> $\sum a_n$ <u>is absolutely convergent</u>.

We observe that the condition (iii) is redundant in view of the fact that $|\bar{N}, p_n|$ and $|R, P_{n-1}, 1|$-summability methods are equivalent, since by (4.1.6), we get :

<u>Theorem C.</u> <u>If</u> (i) $\sum a_n$ <u>is summable</u> $|\bar{N}, p_n|$ <u>and</u> (ii) $\{t_n\} \subset BV$, <u>then</u> $\sum a_n$ <u>is absolutely convergent</u>.

The object of the present chapter is to obtain Tauberian theorems for $|J, p_n|$-summability. We prove these theorems in Sections 4.5 and 4.6.

It is interesting to note that, as a special case of our Theorem 3, we get the following Tauberian theorem of

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1) Ahmad (4), Theorem 2(a); see also Ahmad (1), Chapter VIII, Theorem 2.

2) Bhatt (1).

HYSLOP for absolute Abel summability .

Theorem B.[1] *If* $\Sigma\, a_n$ *is summable* $|A|$ *and* $\{n\, a_n\} \in BV$, *then* $\Sigma\, a_n$ *is absolutely convergent.*

4.3 We establish the following theorems.

Theorem 1. If $\Sigma\, a_n$ *is summable* $|J,\, p_n|$ *and* $\{t_n\} \in BV$, *and if* $\{p_n\}$ *is such that*

(i) $n\, p_n / P_{n-1} < C$, *for* $n = 1, 2, \ldots$

and

(ii) [illegible]$(w) > 0$ *for* $w \geq 1$,

then $\Sigma\, a_n$ *is summable* $|\bar{N},\, p_n|$.

Theorem 2. If $\Sigma\, a_n$ *is summable* $|J,\, p_n|$, *and* $\{t_n\} \in BV$, *and if* $\{p_n\}$ *satisfies the same conditions as in Theorem 1* , *then* $\Sigma\, a_n$ *is absolutely convergent.*

Theorem 3. If $\Sigma\, a_n$ *is summable* $|J,\, p_n|$, *and* $\{a_n\, P_{n-1} / p_n\} \in BV$, *and if* $\{p_n\}$ *satisfies the same conditions as in Theorem 1, then* $\Sigma\, a_n$ *is absolutely convergent.*

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1) Hyslop (1), Theorem 3.

4.4 We require the following lemmas for the proof of our theorems.

__Lemma 1.__[1] __Let__ $u_m > 0$, $\lambda_m = \sum_{n=1}^{m} u_n$, __and__

$$d_m = \lambda_m^{-1} \sum_{n=1}^{m} u_n c_n .$$

__Then, if__ $\{c_n\} \in BV$, $\{d_m\} \in BV$.

__Lemma 2. If__ $\{t_n\} \in BV$ __and__

$$J(s) = \frac{\sum_{n=0}^{\infty} p_n s_n e^{-ns}}{\sum_{n=0}^{\infty} p_n e^{-ns}} \qquad (s > 0),$$

__then__

$$J'(s) = - \frac{\sum_{n=1}^{\infty} t_n \frac{p_n}{P_n} \sum_{v=n}^{\infty} e^{-vs} \sum_{u=n}^{v} (2u-v) P_u p_{v-u}}{\left(\sum_{0}^{\infty} p_n e^{-ns} \right)^2} .$$

__Proof.__ Since, by (4.1.4), (4.1.5) and (4.1.6),

$$t_n = (P_n)^{-1} \sum_{v=1}^{n} P_{v-1} a_v , \quad t_0 = 0 ,$$

and

$$a_n = \frac{p_n}{P_{n-1}} t_n + (t_n - t_{n-1}) ,$$

---

1) Mohanty (1) .

for $n \geq 1$,

$$s_n = a_0 + t_n + \sum_{v=1}^{n} \frac{p_v}{P_{v-1}}\, t_v \,.$$

Hence

$$(4.4.1) \qquad \sum_{n=0}^{\infty} p_n s_n e^{-ns} = a_0 \sum_{n=0}^{\infty} p_n e^{-ns} + \sum_{n=1}^{\infty} t_n e^{-ns} + \sum_{n=1}^{\infty} p_n e^{-ns} \sum_{v=1}^{n} \frac{p_v}{P_{v-1}}\, t_v$$

$$= a_0 \sum_{n=0}^{\infty} p_n e^{-ns} + \sum_{n=1}^{\infty} t_n e^{-ns} + \sum_{v=1}^{\infty} \frac{p_v}{P_{v-1}}\, t_v \sum_{n=v}^{\infty} p_n e^{-ns} \,,$$

the inversion in the last sum being justified by absolute convergence whenever $\{t_v\} \in B$ (and thus, a fortiori, when $\{t_v\} \in BV$).

Also, for $n = 1, 2, 3, \ldots$,

$$(4.4.2) \qquad \left(\sum_{v=n}^{\infty} p_v e^{-vs}\right)\left(\sum_{v=0}^{\infty} e^{-vs}\right)$$

$$= \left(\sum_{v=0}^{\infty} p_{n+v} e^{-(n+v)s}\right)\left(\sum_{v=0}^{\infty} e^{-vs}\right)$$

$$= e^{-ns}\left(\sum_{v=0}^{\infty} p_{n+v} e^{-vs}\right)\left(\sum_{v=0}^{\infty} e^{-vs}\right)$$

$$= e^{-ns} \sum_{v=0}^{\infty} e^{-vs}\left(\sum_{u=0}^{v} p_{n+u}\right)$$

$$= e^{-ns} \sum_{v=0}^{\infty} e^{-vs} (P_{n+v} - P_{n-1})$$

$$= e^{-ns} \sum_{v=0}^{\infty} e^{-vs} P_{n+v} - P_{n-1} e^{-ns}\left(\sum_{v=0}^{\infty} e^{-vs}\right)$$

$$= \sum_{v=0}^{\infty} P_{n+v} e^{-(n+v)s} - P_{n-1} e^{-ns} (1 - e^{-s})^{-1}$$

$$= \sum_{v=n}^{\infty} P_v e^{-vs} - P_{n-1} e^{-ns} (1 - e^{-s})^{-1}$$

Therefore, from (4.4.1) and (4.4.2), for $0 < s \leq \infty$, we have

$$(4.4.3)\quad J(s) = a_0 + \frac{\sum_{n=1}^{\infty} p_n t_n e^{-ns}}{\sum_{n=1}^{\infty} p_n e^{-ns}} + \sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} t_n \sum_{v=n}^{\infty} p_v e^{-vs}$$

$$= a_0 + \frac{\sum_{n=1}^{\infty} p_n t_n e^{-ns}}{\sum_{n=0}^{\infty} p_n e^{-ns}} + \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} t_n \left(\sum_{v=n}^{\infty} p_v e^{-vs}\right)\left(\sum_{v=0}^{\infty} e^{-vs}\right)}{\sum_{n=0}^{\infty} P_n e^{-vs}}$$

$$= a_0 + \frac{\sum_{n=1}^{\infty} p_n t_n e^{-ns}}{\sum_{n=0}^{\infty} p_n e^{-ns}} + \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} t_n \sum_{v=n}^{\infty} P_v e^{-vs}}{\sum_{n=0}^{\infty} P_n e^{-ns}} -$$

$$- \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} P_{n-1} e^{-ns}}{(1-e^{-s}) \sum_{n=0}^{\infty} P_n e^{-ns}}$$

$$= a_0 + \frac{\sum_{n=1}^{\infty} p_n t_n e^{-ns}}{\sum_{n=0}^{\infty} p_n e^{-ns}} + \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} t_n \sum_{v=n}^{\infty} P_v e^{-vs}}{\sum_{n=0}^{\infty} P_n e^{-ns}} -$$

$$- \frac{\sum_{n=1}^{\infty} p_n t_n e^{-ns}}{\sum_{n=0}^{\infty} p_n e^{-ns}}$$

$$= a_0 + \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} t_n \sum_{v=n}^{\infty} e^{-vs} P_v}{\sum_{n=0}^{\infty} P_n e^{-ns}}.$$

Again, since $\{t_n\} \in BV$, the series

$$\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} t_n \sum_{v=n}^{\infty} P_v x^v$$

is absolutely convergent, for then this series is majorised by

$$\sum_{n=1}^{\infty} \sum_{v=n}^{\infty} P_v x^v = \sum_{v=1}^{\infty} v P_v x^v = x \sum_{v=1}^{\infty} v P_v x^{v-1},$$

which converges for $0 \leq x < 1$, by hypothesis.

Therefore, differentiating (4.4.7) with respect to s, we get

$$J'(s) = \frac{-\sum_{n=1}^{\infty} t_n \frac{p_n}{p_{n-1}} \sum_{v=n}^{\infty} e^{-vs} \sum_{u=n}^{v} (2u - v)\, p_u\, p_{v-u}}{\left(\sum_{v=0}^{\infty} p_v\, e^{-vs}\right)^2} .$$

Lemma 2. (i) For $v \geq 0$, $n \geq 0$ ($0 \leq v \leq n$, $n < v \leq 2n$ and $v > 2n$),

$$\sum_{u=0}^{n} (2u - v - 1)\, p_u\, p_{v-u+1} \leq 0 .$$

(ii) For $n \geq 1$ and $v \geq n$,

$$\sum_{u=n}^{v} (2u - v)\, p_u\, p_{v-u} \geq 0 .$$

Proof. (i) is due to McFadden.[1] We give its proof here for completeness. We see that

$$\sum_{u=0}^{n} (2u - v - 1)\, p_u\, p_{v-u+1}$$

$$= \sum_{u=1}^{n} u\, p_u\, p_{v-u+1} - \sum_{u=0}^{n} (v - u + 1)\, p_u\, p_{v-u+1}$$

$$= p_1 p_v + 2 p_2 p_{v-1} + \ldots + n p_n p_{v-n+1} -$$

$$-(v+1) p_0 p_{v+1} - v\, p_1 p_v - (v-1) p_2 p_{v-1} - \ldots - (v-n+1) p_n p_{v-n+1} ,$$

1) McFadden (1), page 181 .

it being understood that $P_r = 0$ when r is negative. If we consider separately the cases $0 \le v \le n$, $n < v \le \rho n$ and $v > \rho n$, it is easily seen that in all cases the given expression is either negative or zero.[1]

(ii) From (i), we infer that, when $n \ge 1$, and for $v \ge 1$,

$$\sum_{u=0}^{n-1} (\rho u - v) P_u P_{v-u} \le 0 .$$

Now

$$\begin{aligned} &\sum_{u=n}^{v} (\rho u - v) P_u P_{v-u} \\ &= \sum_{u=0}^{v} (\rho u - v) P_u P_{v-u} - \sum_{u=0}^{n-1} (\rho u - v) P_u P_{v-u} \\ &= -\sum_{u=0}^{n-1} (\rho u - v) P_u P_{v-u} \ge 0 , \end{aligned}$$

by (i), since $\displaystyle\sum_{u=0}^{v} (\rho u - v) P_u P_{v-u} = 0$.

<u>Lemma 4</u>. <u>Let</u>

$$(4.4.4) \qquad C(w) = \frac{\displaystyle\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v) P_u P_{v-u}}{w\left(\displaystyle\sum_{v=0}^{\infty} P_v e^{-v/w}\right)^2} .$$

---

1) See McFadden (1), page 181.

If [illegible]$(w) > 0$ *for* $w \geq 1$, *then for* $w \geq 1$, $G(w) > 0$.

Proof. Since

$$J_1 < \left\{ w\,(1 - e^{-1/w}) \right\}^{-1} < J_2 \quad (w \geq 1), \tag{4.4.5}$$

where J_1 and J_2 are two suitable positive constants, we observe that

$$G(w) = \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u - v)\, P_u\, P_{v-u}}{w(1 - e^{-1/w}) \left(\sum_{0}^{\infty} e^{-n/w} \right) \left(\sum_{v=0}^{\infty} e^{-v/w} \sum_{u=0}^{v} P_u P_{v-u} \right)}$$

$$> \frac{1}{w(1 - e^{-1/w})} \; \frac{\sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (2u - v)\, P_u\, P_{v-u}}{\sum_{v=0}^{\infty} e^{-v/w} \sum_{u=0}^{v} (v - u + 1)\, P_u\, P_{v-u}}$$

$$= \frac{1}{w(1 - e^{-1/w})} \;[illegible](w)$$

$$> 0 ,$$

by hypothesis and the notation (4.110) . This completes the proof of the lemma .

Lemma 5. If $p_n > 0$ *and* $P_n = p_0 + p_1 + \dots + p_n$, *for* $n = 0, 1, 2, \dots$, *then* $\left\{ p_n / \sum_{v=0}^{\infty} p_v\, e^{-v/n} \right\}$ *is bounded* .

<u>Proof</u>. Since, by hypothesis

$$\sum_{v=0}^{\infty} p_v\, e^{-v/n} \geq \sum_{v=0}^{n} p_v\, e^{-v/n} \geq \frac{1}{e} \sum_{v=0}^{n} p_v = \frac{P_n}{e} ,$$

the lemma follows .

4.5 <u>Proof of Theorem 1</u>. We use the following alternative definition for $|J, p_n|$-summability .

Let

$$J(s) = \frac{\sum_{v=0}^{\infty} p_v\, s_v\, e^{-vs}}{\sum_{v=0}^{\infty} p_v\, e^{-ns}} .$$

Then the sequence $\{s_n\}$, or the series $\Sigma\, a_n$, will be said to be summable $|J, p_n|$ if

$$\int_0^{\infty} |J'(s)|\, ds < \infty .$$

We shall suppose throughout that $m = [w]$, $w = \varepsilon^{-1}$, and that N is a positive integer. By using (4.4.4), it is sufficient to prove that

$$\phi_N = \int_1^N |t_m|\, w^{-1}\, G(w)\, dw = O(1) ,$$

for, by Lemma 4 and (4.1.5),

$$\mathcal{J}_N = \sum_{n=1}^{N-1} \int_n^{n+1} |t_m|\, w^{-1}\, G(w)\, dw$$

$$= \sum_{n=1}^{N-1} |t_n|\, n^{-1} \int_n^{n+1} n\, w^{-1}\, G(w)\, dw$$

$$> \frac{K}{C} \sum_{n=1}^{N-1} \left| \frac{p_n}{P_{n-1}}\, t_n \right| \text{ (by hypothesis (i))}$$

$$= \frac{K}{C} \sum_{n=1}^{N-1} \left| \bar{\Delta} \sigma_n \right| .$$

the result follows.

We write

$$\mathcal{J}_N \leq I_1 + I_2 ,$$

where

$$I_1 = \int_1^N w^{-2} \left| J' \left(\frac{1}{w}\right) \right| dw$$

$$I_2 = \int_1^N \left| w^{-2} J'\left(\frac{1}{w}\right) + t_m\, w^{-1}\, G(w) \right| dw .$$

Now,

$$I_1 = \int_{1/N}^{1} |J'(s)|\, ds = O(1) ,$$

since $\Sigma\, a_n$ is summable $|J, p_n|$. Also, since, by Lemma 1,

$$J'(1/w) = -\frac{\sum_{n=1}^{\infty} t_n \frac{p_n}{P_{n-1}} \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v)\, p_u P_{v-u}}{\left(\sum_{v=0}^{\infty} p_v e^{-v/w}\right)^2},$$

we have

$$\delta_\rho = \int_1^M \frac{\left| \sum_{n=1}^{\infty} \frac{p_n}{P_{n-1}} (t_n - t_n) \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v) p_u P_{v-u} \right|}{w^2 \left(\sum_{v=0}^{\infty} p_v e^{-v/w}\right)^2}\, dw$$

$$\leq \delta_{\rho,1} + \delta_{\rho,2} ,$$

where

$$\delta_{\rho,1} = \int_1^M \frac{\sum_{n=1}^{m-1} \frac{p_n}{P_{n-1}} \sum_{r=n+1}^{m} \left|\bar{\Delta} t_r\right| \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v) p_u P_{v-u}}{w^2 \left(\sum_{v=0}^{\infty} p_v e^{-v/w}\right)^2}\, dw$$

and

$$\delta_{\rho,2} = \int_1^M \frac{\sum_{n=m+1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{r=m+1}^{n} \left|\bar{\Delta} t_r\right| \sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v) p_u P_{v-u}}{w^2 \left(\sum_{v=0}^{\infty} p_v e^{-v/w}\right)^2}\, dw.$$

Now

$$\delta_{\rho,1} = \int_1^M \frac{dw}{w^2} \sum_{n=1}^{m-1} \frac{p_n}{P_{n-1}} \sum_{r=n+1}^{m} \left|\bar{\Delta} t_r\right| \frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (\rho u - v) p_u P_{v-u}}{\left(\sum_{v=0}^{\infty} p_v e^{-v/w}\right)^2}$$

$$= - \int_1^N dw \sum_{n=1}^{m-1} \frac{p_n}{P_{n-1}} \sum_{r=n+1}^{m} \left|\bar{\Delta} t_r\right| \frac{d}{dw}\left[\frac{\sum\limits_{v=0}^{n-1} p_v \, e^{-v/w}}{\sum\limits_{v=0}^{\infty} p_v \, e^{-v/w}} \right]$$

$$= - \sum_{r=2}^{N} \left|\bar{\Delta} t_r\right| \sum_{n=1}^{r-1} \frac{p_n}{P_{n-1}} \int_r^{\infty} \frac{d}{dw}\left[\frac{\sum\limits_{v=0}^{n-1} p_v \, e^{-v/w}}{\sum\limits_{v=0}^{\infty} p_v \, e^{-v/w}} \right] dw$$

$$= \sum_{r=2}^{N} \left|\bar{\Delta} t_r\right| \sum_{n=1}^{r-1} \frac{p_n}{P_{n-1}} \left[- \frac{\sum\limits_{v=0}^{n-1} p_v \, e^{-v/w}}{\sum\limits_{v=0}^{\infty} p_v \, e^{-v/w}} \right]_r^{\infty}$$

$$< \sum_{r=2}^{N} \left|\bar{\Delta} t_r\right| \sum_{n=1}^{r-1} \frac{p_n}{P_{n-1}} \frac{\sum\limits_{v=0}^{n-1} p_v \, e^{-v/r}}{\sum\limits_{v=0}^{\infty} p_v \, e^{-v/r}}$$

$$< \sum_{r=2}^{N} \left|\bar{\Delta} t_r\right| \sum_{n=1}^{r-1} \frac{p_n}{P_{n-1}} P_{n-1} \frac{\sum\limits_{v=0}^{n-1} e^{-v/r}}{\sum\limits_{v=0}^{\infty} p_v \, e^{-v/r}}$$

$$= \sum_{r=2}^{N} \left|\bar{\Delta} t_r\right| \sum_{n=1}^{r-1} \frac{p_n \left(1 - e^{-n/r}\right)}{\left(1 - e^{-1/r}\right) \sum\limits_{v=0}^{\infty} p_v \, e^{-v/r}}$$

$$< \sum_{r=2}^{N} \left|\bar{\Delta} t_r\right| \frac{\sum\limits_{n=0}^{r} p_n}{\sum\limits_{v=0}^{\infty} p_v \, e^{-v/r}}$$

$$= \sum_{r=2}^{M} \left|\bar{\Delta} t_r\right| \frac{P_r}{\sum_{v=0}^{\infty} p_r \, e^{-v/r}}$$

$$\leq K \sum_{r=1}^{M} \left|\bar{\Delta} t_r\right| \qquad \text{(by Lemma 8)}$$

$$\leq K < \infty ,$$

by hypotheses.

Again,

$$S_{2,2} = \int_1^M \frac{dw}{w^2} \sum_{n=m+1}^{\infty} \frac{p_n}{P_{n-1}} \sum_{r=m+1}^{n} \left|\bar{\Delta} t_r\right| \frac{\sum_{v=n}^{\infty} e^{-v/w} \sum_{u=n}^{v} (u-r) p_u P_{v-u}}{\left(\sum_{v=0}^{\infty} P_v \, e^{-v/w}\right)^2}$$

$$= \sum_{r=1}^{M} \left|\bar{\Delta} t_r\right| \sum_{n=r}^{\infty} \frac{p_n}{P_{n-1}} \int_1^r \frac{d}{dw} \left[\frac{\sum_{v=n}^{\infty} P_v \, e^{-v/w}}{\sum_{v=0}^{\infty} P_v \, e^{-v/w}} \right] dw$$

$$= \sum_{r=1}^{M} \left|\bar{\Delta} t_r\right| \sum_{n=r}^{\infty} \frac{p_n}{P_{n-1}} \left[\frac{\sum_{r=n}^{\infty} p_v \, e^{-v/w}}{\sum_{v=0}^{\infty} p_v \, e^{-v/w}} \right]_1^r$$

$$< \sum_{r=1}^{M} \left|\bar{\Delta} t_r\right| \sum_{n=r}^{\infty} \frac{p_n}{P_{n-1}} \frac{\sum_{v=n}^{\infty} P_v \, e^{-v/r}}{\sum_{v=0}^{\infty} P_v \, e^{-v/r}}$$

$$= \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| \frac{\sum\limits_{n=r}^{\infty} \frac{p_n}{P_{n-1}} P_{n-1}\, e^{-n/r}}{(1- e^{-1/r}) \sum\limits_{v=0}^{\infty} p_v\, e^{-v/r}} +$$

$$+ \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| \frac{\sum\limits_{n=r}^{\infty} \frac{p_n}{P_{n-1}} \sum\limits_{v=n}^{\infty} p_v\, e^{-v/r}}{\left(\sum\limits_{n=0}^{\infty} p_n\, e^{-n/r}\right) (1-e^{-1/r})}$$

$$\leq \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| \frac{\sum\limits_{n=r}^{\infty} p_n\, e^{-n/r}}{\sum\limits_{n=0}^{\infty} p_n\, e^{-n/r}} +$$

$$+ \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| \frac{\sum\limits_{v=r}^{\infty} v\, p_v\, e^{-v/r}}{r(1-e^{-1/r})\left(\sum\limits_{v=0}^{\infty} p_v\, e^{-v/r}\right)}$$

$$- \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| \frac{r-1}{r} \frac{\sum\limits_{v=r}^{\infty} p_v\, e^{-v/r}}{\sum\limits_{v=0}^{\infty} p_v\, e^{-v/r}}$$

$$\leq K \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| + K \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right| \frac{\sum\limits_{v=r}^{\infty} p_v\, e^{-v/r}}{r(1-e^{-1/r}) \sum\limits_{v=0}^{\infty} p_v\, e^{-v/r}}$$

$$\leq K \sum_{r=1}^{N} \left|\bar{\Delta} t_r\right|$$

$$\leq K < \infty ,$$

by hypothesis and by the fact that

$$J_1 < \frac{1}{r(1-e^{-1/r})} < J_2 \quad \text{for} \quad r \geq 1 \ ,$$

where J_1 and J_2 are suitable positive constants.

This completes the proof of Theorem 1 .

4.6 <u>Proof of Theorems 2 and 3</u> .

Theorem 2 is obtained by combining the results of Theorem 1 and Theorem C .

Theorem 3 is obtained from Theorem 2, by an appeal to Lemma 1 .

CHAPTER V

ABSOLUTE SUMMABILITY OF POWER SERIES

BY METHODS BASED ON POWER SERIES.

5.1 Definitions and Notations. Suppose that

$$p_n > 0, \sum_{n=0}^{\infty} p_n = \infty ,$$

and that the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n , \quad p(0) = p_0 ,$$

is 1. Given any series $\sum_{n=0}^{\infty} a_n$, with the sequence of partial sums $\{s_n\}$, we shall use the notations :

(5.1.1) $$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n ,$$

and

(5.1.2) $$J(x) \equiv J_s(x) = p_s(x)/ p(x).$$

If the series on the right of (5.1.1) is convergent in the right open interval (0, 1) and if

$$J(x) \in BV\,(c,\ 1)\ ,\quad (\ 0 < c < 1\ ),$$

we say that the series $\sum_{n=0}^{\infty} a_n$, or the sequence $\{s_n\}$, is absolutely summable $(J,\ p_n)$, or simply summable $\left|J,\ p_n\right|$.[1]

In the special cases in which

$$\text{(i)}\ \ p_n = A_n^k = \frac{\Gamma(n+k+1)}{\Gamma(n+1)\ \Gamma(k+1)} \qquad (k > -1;\ n=0,\ 1,\ \ldots.\ )$$

and

$$\text{(ii)}\ \ p_n = (n+1)^{-1}\ ,\quad (\ n= 0,\ 1,\ \ldots.\ )\ ,$$

$\left|J,\ p_n\right|$ - summability reduces respectively to the $\left|A_k\right|$ - summability ($\left|A_0\right|$ being the same as absolute Abel summability $\left|A\right|$) and the absolute logarithmic summability $\left|L\right|$.[2]

Throughout we take $f(x)$ to be an even integrable function with period 2π, and we also write

$$\text{(5.1.3)} \qquad \overset{*}{g}(t) = \int_t^{\pi} \frac{f(u)}{2\sin u/2}\,du\ .$$

5.2. <u>Introduction</u>. Recently, AHMAD[3] discussed a number of problems concerning the method $\left|J,\ p_n\right|$. In this chapter, we propose to apply this method to Fourier series.

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1) Ahmad (1), (3), (4); see also Das (1).

2) Ahmad (1), (4).

3) Ahmad (1), (3), (4).

MOHANTY and PATNAIK [1] have proved the following theorem.

<u>Theorem 2. If</u>

$$(5.2.1) \qquad \frac{g^*(t)}{t \log(2\pi/t)} \in L(0, \pi) ,$$

<u>then the Fourier series of</u> f <u>is</u> $|L|$<u>-summable at the origin</u>.

Recently, IZUMI and IZUMI [2] gave an alternative proof of this theorem. They also generalized this theorem (Theorem 3) for $|J, p_n|$-summability in the following form:

<u>Theorem . Suppose that</u> (i) <u>the sequence</u> $\{n\, p_n\}$ <u>is of bounded variation and that</u> (ii) <u>there is an</u> a, $0 < a < 1$, <u>such that</u>

$$(5.2.2) \qquad (1-x)^{a}\, p(x) \downarrow \quad \text{as} \quad x \uparrow 1 .$$

<u>If</u> $g^*(t) / tp(1-t) \in L(0, \pi)$, <u>then the Fourier series of</u> f <u>is</u> $|J, p_n|$<u>-summable at the origin</u>.

<u>The condition</u> (i) <u>may be replaced by that</u>

$$(5.2.3) \qquad p'(z) = O(1/|1-z|\,), \quad p''(z) = O(1/|1-z|^2)$$

<u>as</u> $z \to 1$, <u>where</u> $z = x\, e^{it}$ <u>and</u> $p(z) = \Sigma\, p_n z^n$.

---

1) Mohanty and Patnaik (1) .

2) Izumi and Izumi (2) .

3) see Izumi and Izumi (2) .

If $p_n = 1/n$ $(n = 1, 2, \ldots)$, then Theorem B reduces to Theorem A.

In the present chapter, our aim is to prove a couple of theorems, one of which (Theorem 1) gives a partial generalization of Theorem B and yields a criterion for absolute Abel summability for Fourier series,[1] while the other gives a complete generalization of Theorem B and contains Theorem A as a special case when $p_n = 1/n$ $(n=1,2,\ldots)$.

5.3 We prove the following theorems :

Theorem 1. Suppose that (i) *the sequence* $\{p_n\}$ *is of bounded variation and that* (ii) *there exists an* a, $1<a<2$, *such that*

$$(5.3.1) \qquad (1-x)^a \, p(x) \downarrow \text{ as } x \uparrow 1 .$$

If $\phi^*(t) / t^2 p(1-t) \in L(0, \pi)$, *then the Fourier series of* f *is* $|J, p_n|$*-summable at the origin.*

The condition (i) *may be replaced by that*

$$(5.3.2) \quad p(z) = O\left(\frac{1}{|1-z|}\right), \quad p'(z) = O\left(\frac{1}{|1-z|^2}\right), \quad p''(z) = O\left(\frac{1}{|1-z|^3}\right)$$

as $z \to 1$, *where* $z = x\,e^{it}$ *and* $p(z) = \sum p_n z^n$.

---

1) For other criteria for absolute Abel summability, see Whittaker (1) and Prasad (1), (2).

Theorem 2. Suppose that (i) *the sequence* $\{p_n\}$ *is of bounded variation* (ii) *there exists an* a, $0 < a < 1$, *such that*

$$(1-x)^{a}\, p(x) \downarrow \text{ as } x \uparrow 1 ,$$

and for $z = x e^{it}$,

$$\text{(iii)} \qquad p'(z) = O(p(z)), \text{ as } z \to 1$$

and

$$\text{(iv)} \qquad (1-z)\, p''(z) = O(p(z)), \text{ as } z \to 1 .$$

If $\overset{*}{g}(t) / tp(1-t) \in L(0, \pi)$, *then the Fourier series of* f *is* $|J, p_n|$*-summable at the origin.*

The condition (i) *may be replaced by that*

$$p(z) = O\left(\frac{1}{|1-z|}\right), \text{ as } z \to 1 .$$

It is interesting to note that we get the following result as a special case of Theorem 1, when

$$p_n = A_n^{k-1} , \qquad -1 < k \leq 0 .$$

Theorem 3. Let $-1 < k \leq 0$. *Suppose that the sequence* $\{p_n\}$ *is of bounded variation. If* $\overset{*}{g}(t) / t^{1-k} \in L(0, \pi)$, *then the Fourier series of* f *is* $|A_k|$*-summable at the origin.*

Remark. In particular, when k= , Theorem 2 gives a criterion for absolute Abel summability.

5.4 Proof of Theorem 1. Let s_n be the n-th partial sum of the Fourier series of f at the origin. Then

$$(5.4.1)\quad \frac{\pi}{2}s_n = \int_0^\pi f(t)\,\frac{\sin(n+1/2)t}{2\sin t/2}\,dt = (n+1/2)\int_0^\pi g^*(t)\cos(n+1/2)t\,dt,$$

where $g(t)$ is defined by (5.1.3).

By (5.1.2) and (5.4.1),

$$\frac{\pi}{2}J(x) = \frac{1}{p(x)}\sum_{n=1}^{\infty}(n+1/2)p_n x^n \int_0^\pi g^*(t)\cos(n+1/2)t\,dt$$

$$= \frac{1}{p(x)}\int_0^\pi g^*(t)\left\{\sum_{n=1}^{\infty}(n+1/2)p_n x^n\cos(n+1/2)t\right\}dt.$$

Putting $p(z) = \sum p_n z^n$ for complex z, we get

$$\sum_{n=1}^{\infty}(n+1/2)p_n x^n\cos(n+1/2)t = R\left\{xe^{it/2}p'(xe^{it}) + \frac{1}{2}e^{it/2}p(xe^{it})\right\}$$

where ' denotes the differentiation with respect to x.

Hence,

$$\int_c^1 |J'(x)|\,dx \le \int_0^\pi |g^*(t)|\,dt \int_c^1 \left|\frac{d}{dx}\left\{\frac{xp'(xe^{it})+\frac{1}{2}p(xe^{it})}{p(x)}\right\}\right|dx,$$

and therefore, it is enough to show that, for $0 < t \le \pi$,

$$(5.4.2)\quad \int_0^1 \left| \frac{d}{dx} \left\{ \frac{xp'(xe^{it}) + \frac{1}{2}p(xe^{it})}{p(x)} \right\} \right| dx = \int_0^1 |\phi(x)|\, dx$$

$$\leq \frac{K}{tp(1-t)},$$

where

$$(5.4.3)\quad \phi(x) = \frac{(1+\frac{1}{2}e^{it})p'(xe^{it}) + xe^{it}\, p''(xe^{it})}{p(x)} -$$

$$- \frac{\{xp'(xe^{it}) + \frac{1}{2}p(xe^{it})\}\, p'(x)}{(p(x))^2}.$$

By the condition (ii), we get

$$\int_0^{1-t} \frac{|p'(xe^{it})|}{p(x)}\, dx \leq \int_0^{1-t} \frac{\sum np_n x^{n-1}}{p(x)}\, dx$$

$$\leq K \int_0^{1-t} \frac{\sum nx^{n-1}}{p(x)}\, dx$$

$$\leq K \int_0^{1-t} \frac{dx}{(1-x)^2\, p(x)}$$

$$\leq \frac{K}{t^a p(1-t)} \int_0^{1-t} \frac{dx}{(1-x)^{3-a}}, \quad (1 < a < 2)$$

$$\leq \frac{K}{t^2 p(1-t)};$$

$$\int_c^{1-t} \frac{|p''(xe^{it})|}{p(x)}\, dx \le K \int_c^{1-t} \frac{dx}{(1-x)^3 p(x)}$$

$$\le \frac{K}{t^a p(1-t)} \int_c^{1-t} \frac{dx}{(1-x)^{3-a}}, \quad (1 < a < 2)$$

$$\le \frac{K}{t^2 p(1-t)};$$

and

$$\int_c^{1-t} \frac{|xp'(xe^{it}) + \frac{1}{2}p(xe^{it})|\, p'(x)}{(\ p(x)\ )^2} \le K \int_c^{1-t} \frac{p'(x)}{(1-x)^2(\ p(x)\ )^2}\, dx$$

$$\le \frac{K}{t^2 p(1-t)}\ .$$

Combining the above three inequalities, we have

(5.4.4) $$\int_c^{1-t} |\lambda(x)|\ dx \le \frac{K}{t^2 p(1-t)}\ .$$

On the other hand, we have

(5.4.5) $$p\ (xe^{it}) = \sum_{n=1}^{\infty} p_n\ x^n\ e^{int}$$

$$= xe^{it} \sum_{n=1}^{\infty} \Delta(p_n)\ \frac{1-x^n\ e^{int}}{1-x\ e^{it}} + \lim_{n\to\infty} p_n\ \frac{xe^{it}}{1-x\ e^{it}}\ ,$$

(5.4.6) $$p'(x\, e^{it}) = \sum_{n=1}^{\infty} n\, p_n\, x^{n-1}\, e^{int}$$

$$= \sum_{n=1}^{\infty} \Delta(p_n) \frac{d}{dx}\left\{ \sum_{m=0}^{n} x^m\, e^{imt} \right\} + \lim_{n\to\infty} p_n \frac{d}{dx}\left(\sum_{m=0}^{\infty} x^m\, e^{imt} \right)$$

$$= \sum_{n=1}^{\infty} \Delta(p_n) \frac{d}{dx}\left\{ \frac{1-x^{n+1}\, e^{i(n+1)t}}{1-x\, e^{it}} \right\} + \lim_{n\to\infty} p_n \frac{e^{it}}{(1-x\, e^{it})^2}$$

$$= \sum_{n=1}^{\infty} \Delta(p_n) \left\{ \frac{1-x^{n+1} e^{i(n+1)t}}{(1-x\, e^{it})^2} - \frac{(n+1)\, x^n e^{i(n+1)t} (1-x\, e^{it})}{(1-x\, e^{it})^2} \right\} +$$

$$+ \lim_{n\to\infty} p_n \frac{e^{it}}{(1-x\, e^{it})^2}$$

and

(5.4.7) $$p''(x\, e^{it}) = \sum_{n=2}^{\infty} n(n-1)\, p_n\, x^{n-2}\, e^{int}$$

$$= \sum_{n=2}^{\infty} \Delta(p_n) \frac{d^2}{dx^2}\left\{ \frac{1-x^{n+1} e^{i(n+1)t}}{1-x\, e^{it}} \right\} + \lim_{n\to\infty} p_n \frac{e^{2it}}{(1-x\, e^{it})^3}$$

$$= \sum_{n=2}^{\infty} \Delta(p_n) \left\{ \frac{2(1-x^{n+1} e^{i(n+1)t})\, e^{2it}}{(1-x\, e^{it})^3} - \frac{2(n+1)x^n e^{i(n+2)t}}{(1-x\, e^{it})^2} - \right.$$

$$\left. - \frac{n(n+1)x^{n-1}\, e^{i(n+1)t}}{(1-x\, e^{it})} \right\} + \lim_{n\to\infty} p_n \frac{e^{2it}}{(1-x\, e^{it})^3}\,.$$

Since p(x) is increasing, we have in view of (5.4.5), (5.4.6) and (5.4.7),

$$(5.4.8)\qquad \int_{1-t}^{1} \frac{|p'(x\,e^{it})|}{p(x)}\,dx \le \frac{1}{p(1-t)} \int_{1-t}^{1} |p'(x\,e^{it})|\,dx$$

$$\le \frac{1}{p(1-t)} \int_{1-t}^{1} \left(\frac{1}{x^2 \sin^2 t} + \frac{1}{x^2 \sin^2 t} \right) dx$$

$$\le \frac{K}{p(1-t)}\,\frac{1}{\sin^2 t}\,\frac{1}{(1-t)^2} \int_{1-t}^{1} dx$$

$$\le \frac{K}{t\,p(1-t)};$$

$$(5.4.9)\qquad \int_{1-t}^{1} \frac{x\,|p''(x\,e^{it})|}{p(x)}\,dx$$

$$\le \frac{1}{p(1-t)} \int_{1-t}^{1} x\,|p''(x\,e^{it})|\,dx$$

$$\le \frac{1}{p(1-t)} \int_{1-t}^{1} x \left\{ \frac{K}{x^3 \sin^3 t} + \frac{K}{x^2 \sin^2 t} + \frac{K}{x \sin t} \right\} dx$$

$$\le \frac{K}{p(1-t)}\,\frac{1}{\sin^3 t}\,\frac{1}{(1-t)^2} \int_{1-t}^{1} dx$$

$$+ \frac{K}{p(1-t)}\,\frac{1}{\sin^2 t}\,\frac{1}{(1-t)} \int_{1-t}^{1} dx + \frac{K}{p(1-t)}\,\frac{1}{\sin t} \int_{1-t}^{1} dx$$

$$\leq \frac{K}{t^2 p(1-t)} + \frac{K}{tp(1-t)} + \frac{K}{p(1-t)}$$

$$\leq \frac{K}{t^2 p(1-t)} ;$$

and

$$(5.4.10) \quad \int_{1-t}^{1} \frac{\left| xp'(x\, e^{it}) + \frac{1}{2} p(x\, e^{it}) \right|}{(\, p(x)\,)^2} p'(x)\, dx$$

$$\leq K \int_{1-t}^{1} \left(\frac{1}{x \sin^2 t} + \frac{1}{\sin t} \right) \frac{p'(x)}{(\, p(x)\,)^2}\, dx$$

$$\leq \frac{K}{\sin^2 t} \frac{1}{(1-t)} \int_{1-t}^{1} \frac{p'(x)}{(p(x))^2}\, dx + \frac{K}{\sin t} \int_{1-t}^{1} \frac{p'(x)}{(p(x))^2}\, dx$$

$$= \frac{K}{\sin^2 t(1-t)} \left[- \frac{1}{p(x)} \right]_{1-t}^{1} + \frac{K}{\sin t} \left[- \frac{1}{p(x)} \right]_{1-t}^{1}$$

$$\leq \frac{K}{t^2 p(1-t)} + \frac{K}{tp(1-t)}$$

$$\leq \frac{K}{t^2 p(1-t)} .$$

Combining the estimates (5.4.8), (5.4.9) and (5.4.10), we get again

$$(5.4.11) \quad \int_{1-t}^{1} \left| \zeta(x) \right|\, dx \leq \frac{K}{t^2 p(1-t)} .$$

Now the inequalities (5.4.4) and (5.4.11) together give the required inequality (5.4.2) . Thus the theorem 1 is established.

5.5 <u>Proof of Theorem 2.</u> As in the proof of Theorem 1, in order to prove Theorem 2, we have to show that, for $0 < t \leq c$,

$$(5.5.1) \qquad \int_c^1 \left| \frac{d}{dx} \left\{ \frac{xp'(x\,e^{it}) + \frac{1}{s}p(x\,e^{it})}{p(x)} \right\} \right| dx = \int_c^1 \left| \omega(x) \right| dx \leq \frac{K}{tp(1-t)} ,$$

where $\omega(x)$ is given by (5.4.3) .

Now, by conditions (i) and (ii), we get

$$(5.5.2) \qquad \int_c^{1-t} \frac{|p'(x\,e^{it})|}{p(x)}\, dx \leq \int_c^{1-t} \frac{\sum np_n x^{n-1}}{p(x)}\, dx$$

$$\leq K \int_c^{1-t} \frac{\sum n\, x^{n-1}}{p(x)}\, dx$$

$$\leq K \int_c^{1-t} \frac{dx}{(1-x)^{\rho} p(x)}$$

$$\leq \frac{K}{t^{a} p(1-t)} \int_c^{1-t} \frac{dx}{(1-x)^{\rho-a}}$$

$$\leq \frac{K}{tp(1-t)} ;$$

$$(5.5.3) \qquad \int_0^{1-t} \frac{|p''(xe^{it})|}{p(x)}\, dx \leq K \int_0^{1-t} \frac{|p(xe^{it})|}{|1-xe^{it}|\, p(x)}\, dx$$

(by condition (iv))

$$\leq K \int_0^{1-t} \frac{1}{(1-x)^2 p(x)}\, dx$$

$$\leq \frac{K}{t^a p(1-t)} \int_0^{1-t} \frac{1}{(1-x)^{2-a}}\, dx$$

$$\leq \frac{K}{tp(1-t)} ;$$

and by conditions (i) and (iii) ,

$$(5.5.4) \qquad \int_0^{1-t} \frac{|xp'(xe^{it}) + \frac{1}{2} p(xe^{it})|\, p'(x)}{(\,p(x)\,)^2}\, dx \leq K \int_0^{1-t} \frac{p'(x)}{(1-x)(p(x))^2}\, dx$$

$$\leq \frac{K}{tp(1-t)} .$$

Combining the above three inequalities, we have

$$(5.5.5) \qquad \int_0^{1-t} |h(x)|\, dx \leq \frac{K}{tp(1-t)} .$$

Since $p(x)$ is increasing, we have in view of (5.4.5) and (5.4.6) ,

$$(5.5.6)\qquad \int_{1-t}^{1} \frac{|p'(xe^{it})|}{p(x)}\,dx \le \frac{1}{p(1-t)} \int_{1-t}^{1} |p'(xe^{it})|\,dx$$

$$\le \frac{K}{p(1-t)} \int_{1-t}^{1} \frac{1}{x^{\rho}\sin^{\rho}t}\,dx$$

$$\le \frac{K}{p(1-t)}\,\frac{1}{\sin^{\rho}t}\,\frac{1}{(1-t)^{2}} \int_{1-t}^{1} dx$$

$$\le \frac{K}{tp(1-t)}\,;$$

$$(5.5.7)\qquad \int_{1-t}^{1} \frac{x\,|p''(xe^{it})|}{p(x)}\,dx \le \frac{1}{p(1-t)} \int_{1-t}^{1} \frac{x\,|p(xe^{it})|}{|1-xe^{it}|}\,dx\,,$$

(by condition (iv))

$$\le \frac{K}{p(1-t)} \int_{1-t}^{1} \frac{x}{x^{\rho}\sin^{\rho}t}\,dx$$

$$\le \frac{K}{p(1-t)}\,\frac{1}{\sin^{\rho}t}\,\frac{1}{(1-t)} \int_{1-t}^{1} dx$$

$$\le \frac{K}{tp(1-t)}\,;$$

and

$$(5.5.8)\qquad \int_{1-t}^{1} \frac{|xp'(xe^{it}) + \frac{1}{2}p(xe^{it})|}{(\,p(x)\,)^{2}}\,p'(x)\,dx$$

$$\leq K \int_{1-t}^{1} (x+\tfrac{1}{2}) \frac{|p(xe^{it})|}{(p(x))^2}\, p'(x)\, dx$$

(by condition (iii))

$$\leq K \int_{1-t}^{1} \frac{1}{\sin t} \frac{p'(x)}{(p(x))^2}\, dx$$

$$\leq \frac{K}{\sin t} \int_{1-t}^{1} \frac{p'(x)}{(p(x))^2}\, dx$$

$$= \frac{K}{\sin t} \left[- \frac{1}{p(x)} \right]_{1-t}^{1}$$

$$\leq \frac{K}{tp(1-t)} .$$

Combining the estimates (5.5.6), (5.5.7) and (5.5.8), we get again

$$(5.5.9) \qquad \int_{1-t}^{1} |f(x)|\, dx \leq \frac{K}{tp(1-t)} .$$

Now, the inequalities (5.5.5) and (5.5.9) together give the required inequality (5.5.1). Thus Theorem 2 is proved.

CHAPTER VI

ON $|J, p_n|_k$ SUMMABILITY OF FOURIER SERIES I

6.1 Definitions and Notations. Suppose that

$$p_n > 0 \,, \quad \sum_{n=0}^{\infty} p_n = \infty \,,$$

and that the radius of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \;; \quad p(0) = p_0$$

is 1. Given any series $\sum_{n=0}^{\infty} a_n$, with the sequence of partial sums $\{s_n\}$, we shall use the notations :

$$p_s(x) = \sum_{n=0}^{\infty} p_n s_n x^n \,, \tag{6.1.1}$$

and

$$J(x) \equiv J_s(x) = p_s(x) \,/\, p(x) \;. \tag{6.1.2}$$

If the series on the right of (6.1.1) is convergent in the right open interval (0, 1) , and if

$$(6.1.3) \qquad \int_c^1 (1-x)^{k-1} \left| \frac{d}{dx}\left\{J(x)\right\} \right|^k dx < \infty,\ 0 < c < 1,\ k \geq 1,$$

then we shall say that the series $\sum_{n=0}^{\infty} a_n$ is summable $|J, p_n|_k$.

It is clear that summability $|J, p_n|_1$ is the same as the summability $|J, p_n|$, defined in Section 5.1 of Chapter V. For $k > 1$, the summability $|J, p_n|$ and $|J, p_n|_k$ are independent of each other.[1] Also for $p_n = 1/n$, $n = 1, 2, \ldots$, we get summability $|L|_k$ [2] and for $p_n = A_n^{\alpha}$ ($\alpha > -1$; $n = 0, 1, \ldots$), it reduces to the well-known method of summability $|A_\alpha|_k$ ($|A_0|_k$ being the same as the summability $|A|_k$).[3]

Throughout we take f to be an even integrable function with period 2π and we write

$$(6.1.4) \qquad \overset{*}{g}(t) = \int_t^{\pi} \frac{f(u)\,du}{2\sin u/2} .$$

Other notations are the same as in Chapter V .

6.2 __Introduction__. MOHANTY and PATNAIK [4] have proved the following theorem.

---

1) cf. Mazhar (1) .

2) Mazhar (1) .

3) Flett (1) .

4) Mohanty and Patnaik (1) .

Theorem A. If the function

$$(6.2.1) \qquad \frac{\overset{*}{g}(t)}{t \log(2\pi/t)} \in L(0, \pi) ,$$

then the Fourier series of f is $|L|$-summable at the origin.

IZUMI and IZUMI[1] gave an alternative proof of this theorem. They also generalized this theorem (Theorem A) for $|J, p_n|$-summability in the following form :

Theorem B. Suppose that (i) the sequence $\{n\, p_n\}$ is of bounded variation and that (ii) there is an a , $0 < a < 1$, such that

$$(6.2.2) \qquad (1-x)^a\, p(x) \downarrow \text{ as } x \uparrow 1 .$$

If $\overset{*}{g}(t)/tp(1-t) \in L(0, \pi)$, then the Fourier series of f is $|J, p_n|$-summable at the origin.

The condition (i) may be replaced by that

$$(6.2.3) \qquad p'(z) = O(1/|1-z|\,),\ p''(z) = O(1/|1-z|^2) \text{ as }$$
$z \to 1$, where $z = xe^{it}$ and $p(z) = \Sigma\, p_n z^n$.

1) Izumi and Izumi (2) .

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If $p_n = 1/n$, then Theorem B reduces to Theorem A .

Generalizing Theorem B, the following theorems were proved in Chapter V of the present Thesis.

<u>Theorem C.</u> <u>Suppose that</u> (i) <u>the sequence</u> $\{p_n\}$ <u>is of bounded variation</u> <u>and that</u> (ii) <u>there exists</u> <u>an</u> a, $1 < a < 2$, <u>such that</u>

$$(1-x)^{a}\, p(x) \downarrow \text{ as } x \uparrow 1 \quad .$$

<u>If</u> $g^{*}(t)/t^{2}p(1-t) \in L(0, \pi)$, <u>then the Fourier series of</u> f <u>is</u> $|J, p_n|$ <u>summable at the origin</u> .

<u>The condition</u> (i) <u>may be replaced by that</u>

$$(6.2.4)\quad p(z) = O\left(\frac{1}{|1-z|}\right),\ p'(z) = O\left(\frac{1}{|1-z|^{2}}\right),\ p''(z) = O\left(\frac{1}{|1-z|^{3}}\right)$$

<u>as</u> $z \to 1$, <u>where</u> $z = xe^{it}$ <u>and</u> $p(z) = \Sigma\, p_n z^n$.

<u>Theorem D.</u> <u>Suppose that</u> (i) <u>the sequence</u> p_n <u>is of bounded variation,</u> (ii) <u>there exists an</u> a, $0 < a < 1$, <u>such that</u>

$$(1-x)^{a}\, p(x) \downarrow \text{ as } x \uparrow 1 \ ,$$

<u>and for</u> $z = xe^{it}$,

(iii) $p'(z) = O(p(z))$, *as* $z \to 1$,

(iv) $(1-z)p''(z) = O(p(z))$, as $z \to 1$.

If $\phi(t) / tp(1-t) \in L(0, \pi)$, *then the Fourier series of* f *is* $|J, p_n|$*-summable at the origin.*

The condition (i) *may be replaced by that*

(6.2.5) $$p(z) = O\left(\frac{1}{|1-z|}\right) \text{ as } z \to 1 .$$

As observed in Chapter V, Theorem C gives a partial generalization of Theorem B and yields a criterion for absolute Abel summability for Fourier series [1] while Theorem D gives a complete generalization of Theorem B and contains Theorem A as a special case when $p_n = 1/n (n=1,2,\ldots)$.

Our object here is to prove a couple of corresponding theorems for summability $|J, p_n|_k$, $k \geq 1$, of Fourier series, which contain Theorems C and D as special cases.

6.3 We establish the following theorems.

Theorem 1. Let $k \geq 1$. *Suppose that* (i) *the sequence* $\{p_n\}$ *is of bounded variation,* (ii) *there is an* a, $1 < a/k < 2$, *such that*

(6.3.1) $$(1-x)^{a/k} p(x) \downarrow \text{ as } x \uparrow 1 ,$$

---

1) For other criteria for absolute Abel summability, see Whittaker (1) and Prasad (1), (2).

and that (iii) *for* $k > 1$, *the function* $\{(1-x)p(x)\}^{1-k}$ *is bounded in* $(c, 1)$.[1] *If* $\overset{*}{g}(t)/t^{2}\, p(1-t) \in L(0,\pi)$, *then the Fourier series of* f *is* $|J, p_n|_k$*-summable at the origin.*

The condition (i) *may be replaced by that*

$$(6.3.9)\quad p(z) = O(1/|1-z|),\ p'(z) = O(1/|1-z|^2),$$

$$p''(z) = O(1/|1-z|^3) \quad \text{as} \quad z \to 1\,,$$

where $z = xe^{it}$ *and* $p(z) = \Sigma\, p_n z^n$.

Theorem 9. Let $k \geq 1$. *Suppose that* (i) *the sequence* $\{p_n\}$ *is of bounded variation,* (ii) *there is an* a, $0 < a/k < 1$, *such that*

$$(1-x)^{a/k}\, p(x) \downarrow \text{ as } x \uparrow 1\,,$$

and for $z = xe^{it}$,

$$\text{(iii)}\qquad p'(z) = O(p(z)),\ \text{as}\ z \to 1\,,$$

and

$$\text{(iv)}\qquad (1-z)\, p''(z) = O(p(z)),\ \text{as}\ z \to 1.$$

If $\overset{*}{g}(t)/tp(1-t) \in L(0, \pi)$, *then the Fourier series of* f *is* $|J, p_n|_k$ *-summable at the origin.*

---

1) For $k = 1$, this condition is void.

The condition (i) *may be replaced by that*

$$(6.?.3) \qquad p(z) = O\left(\frac{1}{|1-z|}\right).$$

It is to note that in the special cases when $p_n = 1$, for $n = 0, 1, 2, \ldots$ and $p_n = \frac{1}{n}$, for $n = 1, 2, \ldots$, we get the following interesting results respectively from our theorems 1 and 2.

Corollary I. Let $k \geq 1$ *and* $-1 < \alpha \leq 0$. *Suppose that the sequence* $\{p_n\}$ *is of bounded variation. If* $\overset{*}{g}(t)/t^{1-\alpha} \in L(0, \pi)$, *then the Fourier series of* f *is* $|A_\alpha|_k$ *-summable at the origin.*

Corollary II. Let $k \geq 1$. *Suppose that the sequence* $\{p_n\}$ *is of bounded variation. If* $\overset{*}{g}(t)/t \log(2\pi/t) \in L(0, \pi)$, *then the Fourier series of* f *is* $|L|_k$ *-summable at the origin.*

6.4 *Proof of Theorem 1.* Since the theorem is known for $k = 1$ (Theorem C), we proceed to prove it for $k > 1$.

Let s_n be the n-th partial sum of the Fourier series of f at the origin. Then

$$(6.4.1) \qquad \frac{\pi}{2} s_n = \int_0^\pi f(t) \frac{\sin(n+\frac{1}{2})t}{2 \sin t/2} dt$$

$$= (n+1/2) \int_0^\pi \overset{*}{g}(t) \cos (n+1/2)t \, dt ,$$

where $\overset{*}{g}(t)$ is defined by (6.1.4); and

$$\frac{1}{\rho} J(x) = \frac{1}{p(x)} \sum_{n=1}^{\infty} (n + {}^{1}/\rho)\, p_n\, x^n \int_0^{\pi} g^*(t) \cos (n+{}^{1}/\rho)t \, dt$$

$$= \frac{1}{p(x)} \int_0^{\pi} g^*(t) \left\{ \sum_{n=1}^{\infty} (n+ {}^{1}/\rho)\, p_n\, x^n \cos(n+ {}^{1}/\rho)\, t \right\} dt.$$

Putting $p(z) = \sum p_n z^n$, for complex z, we get

$$\sum_{n=1}^{\infty} (n+{}^{1}/\rho) p_n x^n \cos(n+{}^{1}/\rho)t = R\left[x e^{it/\rho} p'(xe^{it}) + \frac{1}{\rho} e^{it/\rho} p(xe^{it})\right]$$

where ' denotes the differentiation with respect to x .

Hence,

$$|J'(x)|\, dx \le \frac{\rho}{\pi} \int_0^{\pi} |g^*(t)|\, |F'(x,t)|\, dt$$

where

(6.4.2) $$F'(x,t) = \frac{d}{dx} \left\{ \frac{xp'(xe^{it}) + \frac{1}{\rho}\, p(xe^{it})}{p(x)} \right\} .$$

Now, the Fourier series of f is summable $|J, p_n|_k$, if

(6.4.3) $$I = \int_0^1 (1-x)^{k-1}\, |J'(x)|^k\, dx < \infty .$$

We see that

$$I \le \int_0^1 (1-x)^{k-1} \left\{ \frac{\rho}{\pi} \int_0^{\pi} |g^*(t)|\, |F'(x,t)|\, dt \right\}^k dx$$

$$= \int_0^1 \left\{ \frac{2}{\pi} \int_0^\pi (1-x)^{1-\frac{1}{k}} \left|g^*(t)\right| \, \left|F'(x,t)\right| dt \right\}^k dx$$

$$\le \left[\frac{2}{\pi} \int_0^\pi \left\{ \int_0^1 (1-x)^{k-1} \left|g^*(t)\right|^k \left|F'(x,t)\right|^k dx \right\}^{1/k} dt \right]^k$$

$$= \left[\frac{2}{\pi} \int_0^\pi \left|g^*(t)\right| \left\{ \int_0^1 (1-x)^{k-1} \left|F'(x,t)\right|^k dx \right\}^{1/k} dt \right]^k ,$$

by using the MINKOWSKI's inequality for integrals.[1]

Therefore, in view of the hypothesis, it is sufficient to prove that

$$(6.4.4) \quad \mathcal{J}(t) \equiv \int_0^1 (1-x)^{k-1} \left|F'(x,t)\right|^k dx \le \frac{K}{t^{2k}\, p^k(1-t)} .$$

Now,

$$\int_0^1 (1-x)^{k-1} \left|F'(x,t)\right|^k dx$$

$$= \int_0^{1-t} (1-x)^{k-1} \left|F'(x,t)\right|^k dx + \int_{1-t}^1 (1-x)^{k-1} \left|F'(x,t)\right|^k dx$$

$$= \mathcal{J}_1(t) + \mathcal{J}_2(t) \text{ , say .}$$

Since

$$(6.4.5) \qquad F'(x,t) = \frac{d}{dx} \left\{ \frac{xp'(xe^{it}) + \frac{1}{2} p(xe^{it})}{p(x)} \right\}$$

---

1) See Hardy, Littlewood and Pólya (1), inequality (6.13.9), p.148.

$$= \frac{p(x)\left[\left\{xp''(xe^{it}).e^{it}+p'(xe^{it})\right\}+\frac{1}{\rho}p'(xe^{it}).e^{it}\right]}{(\ p(x)\ )^{\rho}}$$

$$- \frac{\left\{xp'(xe^{it})+\frac{1}{\rho}\ p(xe^{it})\right\}\ p'(x)}{(\ p(x)\ )^{2}}$$

$$= \frac{(1+e^{it}/\rho)p'(xe^{it})\ +\ xe^{it}\ p''\ (xe^{it})}{p(x)}\ -$$

$$- \frac{\left\{xp'(xe^{it})\ +\ \frac{1}{\rho}p\ (xe^{it})\right\}\ p'(x)}{(\ p(x)\ )^{\rho}}\ ,$$

by virtue of the condition (ii), we get

$$(6.4.6)\quad \mathfrak{J}_1(t)\ \leq K\int_c^{1-t}(1-x)^{k-1}\left[\frac{|p'(xe^{it})|}{p(x)}+\frac{|p''(xe^{it})|}{p(x)}+\right.$$

$$\left.+\frac{|p'(xe^{it})\ +\frac{1}{\rho}\ p(xe^{it})|\ p'(x)}{(\ p(x)\ )^{\rho}}\right]^{k}dx$$

$$\leq K\int_c^{1-t}(1-x)^{k-1}\left[\frac{1}{(1-x)^{\rho}p(x)}+\frac{1}{(1-x)^{3}p(x)}+\frac{1}{(1-x)^{\rho}(p(x))^{\rho}}\right]^{k}dx$$

$$\leq K\int_c^{1-t}(1-x)^{k-1}\left\{\frac{1}{(1-x)^{3}\ p(x)}\right\}^{k}\ dx$$

$$\leq K\int_c^{1-t}\frac{1}{(1-x)^{a}\ p^{k}(x)}\cdot\frac{1}{(1-x)^{\rho k+1-a}}\ dx$$

$$\leq \frac{A}{t^{a} p^{k}(1-t)} \int_{c}^{1-t} \frac{1}{(1-x)^{2k+1-a}}\, dx$$

$$\leq \frac{A}{t^{2k} p^{k} (1-t)} .$$

o find an estimate for $\mathfrak{J}_p(t)$, as in Chapter V of the present Thesis, we write

$$(6.4.7)\quad p(xe^{it}) = xe^{it} \sum_{n=1}^{\infty} \Delta(p_n) \frac{1-x^n e^{int}}{1-x e^{it}} + \lim_{n\to\infty} p_n \frac{xe^{it}}{1-x e^{it}},$$

$$(6.4.8)\quad p'(xe^{it}) = \sum_{n=1}^{\infty} \Delta(p_n) \left\{ \frac{1-x^{n+1} e^{i(n+1)t}}{(1-x e^{it})^2} - \frac{(n+1)x^n e^{i(n+1)t}(1-xe^{it})}{(1-xe^{it})^2} \right\} + \lim_{n\to\infty} p_n \frac{e^{it}}{(1-xe^{it})^2},$$

and

$$(6.4.9)\quad p''(xe^{it}) = \sum_{n=2}^{\infty} \Delta(p_n) \left\{ \frac{2(1-x^{n+1} e^{i(n+1)t})\, e^{2it}}{(1-xe^{it})^3} - \frac{2(n+1)x^n e^{i(n+2)t}}{(1-xe^{it})^2} - \frac{n(n+1)x^{n-1} e^{i(n+1)t}}{(1-xe^{it})} \right\} +$$

$$+ \lim_{n\to\infty} p_n \frac{e^{2it}}{(1-xe^{it})^3} .$$

ince $p(x)$ is increasing, we have, in view of (6.4.7), (6.4.8) and (6.4.9),

$$(6.4.10) \quad \int_{1-t}^{1} (1-x)^{k-1} \frac{|p'(xe^{it})|^{k}}{p^{k}(x)}\, dx$$

$$\leq \frac{K}{p^{k}(1-t)} \int_{1-t}^{1} (1-x)^{k-1} \left\{ \frac{1}{x^{\rho}\sin^{\rho} t} \right\}^{k} dx$$

$$\leq \frac{K}{p^{k}(1-t)} \frac{1}{(1-t)^{\rho k}\sin^{\rho k} t} \int_{1-t}^{1} (1-x)^{k-1}\, dx$$

$$= \frac{K}{p^{k}(1-t)} \frac{1}{(1-t)^{\rho k}\sin^{\rho k} t} \left[-\frac{(1-x)^{k}}{k} \right]_{1-t}^{1}$$

$$\leq \frac{K}{p^{k}(1-t)} \frac{t^{k}}{(1-t)^{\rho k}\sin^{\rho k} t}$$

$$\leq \frac{K}{t^{k}p^{k}(1-t)} \,;$$

$$(6.4.11) \quad \int_{1-t}^{1} (1-x)^{k-1} \frac{x^{k}\,|p''(xe^{it})|^{k}}{p^{k}(x)}\, dx$$

$$\leq \frac{K}{p^{k}(1-t)} \int_{1-t}^{1} (1-x)^{k-1} x^{k} \left| p''(xe^{it}) \right|^{k} dx$$

$$\leq \frac{K}{p^k(1-t)} \int_{1-t}^{1} (1-x)^{k-1} x^k \left\{ \frac{1}{x^{3k}\sin^{3k}t} + \frac{1}{x^{2k}\sin^{2k}t} + \frac{1}{x^k \sin^k t} \right\} dx$$

$$\leq \frac{K}{p^k(1-t)} \frac{1}{(1-t)^{2k}\sin^{3k}t} \int_{1-t}^{1} (1-x)^{k-1}\, dx \; +$$

$$+ \frac{K}{p^k(1-t)} \frac{1}{(1-t)^k \sin^{2k}t} \int_{1-t}^{1} (1-x)^{k-1}\, dx \; +$$

$$+ \frac{K}{p^k(1-t)} \frac{1}{\sin^k t} \int_{1-t}^{1} (1-x)^{k-1}\, dx$$

$$\leq \frac{Kt^k}{p^k(1-t)(1-t)^{2k}\sin^{3k}t} + \frac{Kt^k}{p^k(1-t)(1-t)^k \sin^{2k}t} + \frac{Kt^k}{p^k(1-t)\sin^k t}$$

$$\leq \frac{K}{t^{2k}\, p^k(1-t)} + \frac{K}{t^k\, p^k(1-t)} + \frac{K}{p^k(1-t)}$$

$$\leq \frac{K}{t^{2k}\, p^k(1-t)} \;;$$

and when $\left\{ (1-x)p(x) \right\}^{1-k}$ is bounded in $(c, 1)$, we have

$$(6.4.12) \qquad \int_{1-t}^{1} (1-x)^{k-1} \frac{\left| xp'(xe^{it}) + \frac{1}{2}\, p(xe^{it}) \right|^k (p'(x))^k}{(p(x))^{2k}}\, dx$$

$$\leq K \int_{1-t}^{1} (1-x)^{k-1} \left\{ \frac{1}{x^k \sin^{2k} t} + \frac{1}{\sin^k t} \right\} \frac{1}{(1-x)^{2(k-1)}} \frac{p'(x)}{(p(x))^{2k}} \, dx$$

$$= K \int_{1-t}^{1} \frac{1}{(1-x)^{k-1}(p(x))^{k-1}} \left\{ \frac{1}{x^k \sin^{2k} t} + \frac{1}{\sin^k t} \right\} \frac{p'(x)}{(p(x))^{k+1}} \, dx$$

$$\leq K \left\{ \frac{1}{(1-t)^k \sin^{2k} t} + \frac{1}{\sin^k t} \right\} \int_{1-t}^{1} \frac{p'(x)}{(p(x))^{k+1}} \, dx$$

$$\leq K \left(\frac{1}{t^{2k}} + \frac{1}{t^k} \right) \left[-\frac{1}{(p(x))^k} \right]_{1-t}^{1}$$

$$\leq \frac{K}{t^{2k} \, p^k(1-t)} \, .$$

Combining the estimates (6.4.10), (6.4.11) and (6.4.12), we get

$$(6.4.13) \quad \mathfrak{J}_p(t) = \int_{1-t}^{1} (1-x)^{k-1} \left| F'(x,t) \right|^k dx \leq \frac{K}{t^{2k} p^k(1-t)} \, .$$

Now the inequalities (6.4.6) and (6.4.13) together give the required inequality (6.4.4) . Thus Theorem 1 is established.

6.5 __Proof of Theorem 2.__ Here also, since the theorem is known for k=1 (Theorem D), we proceed to prove it for k > 1.

As in the proof of Theorem 1, in order to prove Theorem 2, we have to prove that

$$(6.5.1) \qquad \mathfrak{J}(t) = \int_0^1 (1-x)^{k-1} \left| P'(x,t) \right|^k dx \le \frac{K}{t^k p^k(1-t)},$$

where $P'(x,t)$ is given by (6.4.2) .

By (6.4.5) and by the virtue of the conditions (ii), (iii) and (iv), we get

$$(6.5.2) \qquad \int_0^{1-t} (1-x)^{k-1} \left| P'(x,t) \right|^k dx$$

$$\le K \int_0^{1-t} (1-x)^{k-1} \left[\frac{|p'(xe^{it})|}{p(x)} + \frac{|p''(xe^{it})|}{p(x)} + \frac{|p'(xe^{it}) + \frac{1}{\rho} p(xe^{it})| \; p'(x)}{(p(x))^{\rho}} \right]^k dx$$

$$\le K \int_0^{1-t} (1-x)^{k-1} \left[\frac{1}{(1-x)^{\rho} p(x)} + \frac{1}{(1-x)^{\rho} p(x)} + \frac{p'(x)}{(1-x)(p(x))^{\rho}} \right]^k dx$$

$$\le K \int_0^{1-t} (1-x)^{k-1} \left\{ \frac{1}{(1-x)^{\rho} p(x)} \right\}^k dx$$

$$\le K \int_0^{1-t} \frac{1}{(1-x)^{a} p^k(x)} \cdot \frac{1}{(1-x)^{k+1-a}} dx$$

$$\leq \frac{K}{t^{a}\, p^{k}(1-t)} \int_{0}^{1-t} \frac{1}{(1-x)^{k+1-a}}$$

$$\leq \frac{K}{t^{k}\, p^{k}(1-t)} .$$

Also, since $p(x)$ is increasing, we have in view of (6.4.7), (6.4.8) and (6.4.) ,

$$(6.5.3) \qquad \int_{1-t}^{1} (1-x)^{k-1} \frac{|p'(xe^{it})|^{k}}{p^{k}(x)}\, dx$$

$$\leq \frac{K}{p^{k}(1-t)} \int_{1-t}^{1} (1-x)^{k-1} \left\{ \frac{1}{x^{2}\sin^{2} t} \right\}^{k} dx$$

$$\leq \frac{K}{p^{k}(1-t)} \frac{1}{(1-t)^{2k}\sin^{2k} t} \int_{1-t}^{1} (1-x)^{k-1}\, dx$$

$$= \frac{K}{p^{k}(1-t)} \frac{1}{(1-t)^{2k}\sin^{2k} t} \left[- \frac{(1-x)^{k}}{k} \right]_{1-t}^{1}$$

$$\leq \frac{K}{p^{k}(1-t)} \frac{t^{k}}{(1-t)^{2k}\sin^{2k} t}$$

$$\leq \frac{K}{t^{k} p^{k}(1-t)} ;$$

$$(6.5.4) \qquad \int_{1-t}^{1} (1-x)^{k-1} \frac{x^{k} \,|p''(xe^{it})|^{k}}{p^{k}(x)}\, dx$$

$$\leq \frac{K}{p^k\,(1-t)} \int_{1-t}^{1} (1-x)^{k-1} \frac{x^k\,|p(xe^{it})|^k}{|1-x\,e^{it}|} \, dx$$

$$\leq \frac{K}{p^k\,(1-t)} \int_{1-t}^{1} (1-x)^{k-1}\, x^k \frac{1}{x^{2k}\sin^{2k} t} \, dx$$

$$\leq \frac{K}{p^k\,(1-t)} \frac{1}{(1-t)^k \sin^{2k} t} \int_{1-t}^{1} (1-x)^{k-1} \, dx$$

$$\leq \frac{K}{p^k\,(1-t)} \frac{t^k}{(1-t)^k \sin^{2k} t}$$

$$\leq \frac{K}{t^k\, p^k\,(1-t)} \;;$$

and

$$(6.5.6) \qquad \int_{1-t}^{1} (1-x)^{k-1} \frac{\left|xp'(xe^{it}) + \frac{1}{\rho}\, p(xe^{it})\right|^k (p'(x))^k}{(p(x))^{2k}} \, dx$$

$$\leq K \int_{1-t}^{1} (1-x)^{k-1} \frac{(x+\frac{1}{\rho})\, |p(xe^{it})|^k}{(p(x))^{2k}} \,(p'(x))^k \, dx$$

$$\leq \int_{1-t}^{1} \frac{1}{\sin^k t} \frac{p'(x)}{(p(x))^{2k}} \, dx$$

$$\leq \frac{K}{\sin^k (t)} \left[- \frac{1}{(p(x))^{2k-1}} \right]_{1-t}^{1}$$

$$\leq \frac{K}{t^k (p(1-t))^{2k-1}}$$

$$\leq \frac{K}{t^k p^k (1-t)}$$

Combining the estimates (6.5.3), (6.5.4) and (6.5.5), we obtain

$$(6.5.6) \qquad \int_{1-t}^{1} (1-x)^{k-1} \left| \phi'(x,t) \right|^k dx \leq \frac{K}{t^k\, p^k(1-t)} .$$

Now the inequalities (6.5.2) and (6.5.6) together give the required inequality (6.5.1) and complete the proof of Theorem 9.

CHAPTER VII

ON $|J, p_n|_k$ -SUMMABILITY OF FOURIER SERIES II*

7.1 <u>Definitions and Notations</u>. We adopt the definition of summability $|J, p_n|_k$, $k \geq 1$, and other notations as given in Section 6.1 of Chapter VI.

Throughout we write

$$(7.1.1) \qquad g(t) = \int_t^{\pi} \frac{\phi(u)}{2\sin u/2}\, du \; ;$$

where

$$\phi(u) = \frac{1}{2}\{f(x_0 + u) + f(x_0 - u) - 2s\}.$$

7.2 <u>Introduction</u>. MOHANTY and PATNAIK [1] have proved the following theorem for an even function f.

<u>Theorem A. If the function</u>

$$(7.2.1) \qquad \frac{g(t)}{t\log(2\pi/t)} \in L(0, \pi),$$

---

* Rahiman (1).

1) Mohanty and Patnaik (1).

then the Fourier series of f is $|L|$-summable at the point x_0.

Recently IZUMI and IZUMI [1] gave an alternative proof of this theorem. Generalizing this, they [2] have also proved the following theorem :

Theorem B. Suppose that (i) $\{n\, p_n\}$ *and* $\{n^2 p_n\}$ *are monotone and concave or convex and that*

$$\text{(ii)} \qquad (1-x)^2\, p''(x)/\, p(x) \in L(0,1).$$

If

$$(7.1.2) \qquad \int_0^1 G(t)\, t^{-3} dt \int_{1-t}^1 ((1-x)^2 p''(x)/p(x))\, dx < \infty ,$$

where $G(t) = \int_0^t |g(u)|\, du$, *then the Fourier series of* f *is* $|J, p_n|$*-summable at the point* x_0 .

Our object here is to establish a theorem for summability $|J, p_n|_k$, $k \geq 1$, of Fourier series, which contains Theorem B as a special case when $k = 1$.

7.2 We establish the following theorem.

---

1) Izumi and Izumi (2) .

2) Izumi and Izumi (1) .

<u>Theorem</u>. <u>Suppose that</u> (i) $\{n\, p_n\}$ <u>and</u> $\{n^2 p_n\}$ <u>are monotone and concave or convex and that</u> (ii) $(1-x)^{3-1/k}\, p''(x) / p(x) \in L^k(0, 1)$. <u>If</u>

$$(7.3.1) \qquad \int_0^1 \frac{G^k(t)}{t^{2k+1}}\, dt \int_{1-t}^1 (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)}\right)^k dx < \infty$$

<u>where</u> $G(t) = \left(\int_0^t |g(u)|^k\, du \right)^{1/k}$, <u>then the Fourier series of</u> f <u>is</u> $|J, p_n|_k$<u>- summable at the point</u> x_0 .

<u>Remark</u>. The condition (7.3.1) is satisfied when

$$(7.3.2) \qquad \left(\int_0^t u^{3k-1} \left(\frac{p''(1-u)}{p(1-u)}\right)^k du \right)^{1/k} \leq t^3\, \frac{p''(1-t)}{p(1-t)} ,$$

for all $t > 0$, and

$$(7.3.3) \qquad t^{1-1/k}\, \frac{p''(1-t)}{p(1-t)}\, G(t) \in L^k(0, 1).$$

7.4 For the proof of our theorem, we need the following lemma.

<u>Lemma</u>. <u>For</u> $0 < c < 1$,

$$\left| \sum_{n=1}^{\infty} n\, p_n \cos(n+1/2)t\, (x^n/p(x))' \right| \leq \frac{K(1-x)^2}{(1-x)^2 + t^2} \cdot \frac{p''(x)}{p(x)}$$

__on the interval__ $(c, 1)$, __where K is a constant.__

The proof of this lemma is contained in the proof of Theorem 2 of IZUMI and IZUMI.[1]

7.5 __Proof of the Theorem.__ We can suppose that

$$\int_0^{\pi} \phi(u)\, du = 0 \quad \text{and} \quad p_1 = p_2 = 0 .$$

The sequence $\left\{n\, p_n;\ n \geq 3\right\}$ is also monotone and concave or convex. Let $s_n(x_o)$ be the n-th partial sum of the Fourier series of f at the point x_o. Then

$$s_n(x_o) = \frac{1}{\pi} \int_0^{\pi} \phi(t) \quad \frac{\sin(n+1/2)t}{2\sin t/2}\, dt.$$

Therefore,

$$J(x) = \frac{1}{p(x)} \sum_{n=1}^{\infty} p_n\, s_n\, (x_o)\ x^n$$

$$= \frac{1}{\pi p(x)} \int_0^{\pi} \frac{\phi(t)}{2\sin t/2} \left(\sum_{n=1}^{\infty} p_n \sin(n+1/2)t\ x^n \right) dt.$$

Differentiating with respect to x, we get

$$J'(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\phi(t)}{2\sin t/2} \left(\sum_{n=1}^{\infty} p_n \sin(n+1/2)t\ (x^n/p(x))' \right) dt$$

---

1) Izumi and Izumi (1), pages 650-651 .

$$(7.5.1) \qquad = \frac{1}{\pi}\int_0^\pi g(t)\Big(\sum_{n=1}^{\infty}(n+{}^1/2)p_n\cos(n+{}^1/2)t\,(x^n/p(x))'\Big)\,dt,$$

where ' denotes the differentiation with respect to x.
Let us write

$$J'(x) = \frac{1}{\pi}\int_0^\pi g(t)\; F'(x,\ t)\ dt,$$

where

$$F'(x,t) = \sum_{n=1}^{\infty}(n+{}^1/2)\ p_n\ \cos(n+{}^1/2)t\ (x^n/\ p(x))'\ .$$

Now, the Fourier series of f is summable $|J,\ p_n|_k$, if

$$(7.5.2) \qquad \int_0^1 (1-x)^{k-1}\,\big|\,J'(x)\big|^k\ dx < \infty\ .$$

Since,

$$\begin{aligned} J'(x) &= \frac{1}{\pi}\int_0^\pi g(t)\ F'(x,t)\ dt \\ &= \frac{1}{\pi}\int_0^\pi g(t)\ F_1'\ (x,\ t)\,dt + \frac{1}{\pi}\int_0^\pi g(t)\,F_2'(x,t)\ dt \\ &= J_1'(x) + J_2'(x)\ , \end{aligned}$$

say, where

$$J_1'(x,t) = \sum_{n=1}^{\infty} n\, p_n \cos(n+ 1/p)t\, (x^n/p(x))'$$

and

$$J_o'(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} p_n \cos (n+ 1/p)t\, (x^n/p(x))' .$$

In order to prove (7.5.2), by HÖLDER's inequality, it is enough to show that

(7.5.3) $$I_r = \int_0^1 (1-x)^{k-1} \left| J_r'(x) \right|^k dx < \infty , \quad (r = 1, 0).$$

Proof of (7.5.3). We have

$$I_1 = \int_0^1 (1-x)^{k-1} \left| J_1'(x) \right|^k dx$$

$$\leq \int_0^1 (1-x)^{k-1} \left(\frac{1}{\pi} \int_0^\pi |g(t)| \; |F_1'(x,t)| \; dt \right)^k dx$$

$$\leq \left(\frac{1}{\pi} \right)^k \int_0^1 dx \int_0^\pi (1-x)^{k-1} |g(t)|^k |F_1'(x,t)|^k dt \left(\int_0^\pi dt\right)^{k-1}$$

$$= \frac{\pi^{k-1}}{\pi^k} \int_0^1 dx \int_0^\pi (1-x)^{k-1} |g(t)|^k \; |F_1'(x,t)|^k dt$$

$$\leq \frac{1}{\pi} \int_0^\pi |g(t)|^k dt \int_0^1 (1-x)^{k-1} \; |F_1'(x,t)|^k dx$$

$$= \frac{1}{\pi}\int_0^{1-c}|g(t)|^k\,dt\int_0^{1-t}(1-x)^{k-1}\left|W_1'(x,t)\right|^k dx$$

$$+\frac{1}{\pi}\int_0^{1-c}|g(t)|^k\,dt\int_{1-t}^{1}(1-x)^{k-1}\left|W_1'(x,t)\right|^k dx$$

$$+\frac{1}{\pi}\int_{1-c}^{\pi}|g(t)|^k\,dt\int_0^{1}(1-x)^{k-1}\left|W_1'(x,t)\right|^k dx$$

$$= I_{11} + I_{12} + I_{13}, \text{ say.}$$

Now, we see that

$$(7.5.4)\quad I_{11} = \frac{1}{\pi}\int_0^{1-c}|g(t)|^k dt\int_0^{1-t}(1-x)^{k-1}\left|W_1'(x,t)\right|^k dx$$

$$\le \frac{K}{\pi}\int_0^{1-c}|g(t)|^k dt\int_0^{1-t}(1-x)^{k-1}\left(\frac{p'(x)}{p(x)}\right)^k dx$$

(by the lemma)

$$= K\int_0^{1}(1-x)^{k-1}\left(\frac{p'(x)}{p(x)}\right)^k dx\left(\int_0^{1-x}|g(t)|^k dt\right)^{1/k\cdot k}$$

$$= K\int_0^{1}(1-x)^{k-1}\left(\frac{p'(x)}{p(x)}\right)^k\left(G(1-x)\right)^k dx$$

$$= -K\int_{1-c}^{0}t^{k-1}\left(\frac{p'(1-t)}{p(1-t)}\right)^k\left(G(t)\right)^k dt$$

$$= K \int_0^{1-c} t^{k-1} \left(\frac{p''(1-t)}{p(1-t)} \right)^k \left(G(t) \right)^k dt$$

$$\leq K \int_0^{1} t^{k-1} \left(\frac{p''(1-t)}{p(1-t)} \right)^k \left(G(t) \right)^k dt$$

$$\leq \varepsilon \, .$$

by (7.3.3) .

Next,

$$I_{12} = \frac{1}{\pi} \int_0^{1-c} \left| g(t) \right|^k dt \int_{1-t}^{1} (1-x)^{k-1} \left| g_1'(x,t) \right|^k dx \tag{7.6.6}$$

$$\leq \frac{K}{\pi} \int_0^{1-c} \frac{|g(t)|}{t^{2k}} \, dt \int_{1-t}^{1} (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)} \right)^k dx$$

(by the lemma)

$$= K \int_0^{1} (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)} \right)^k dx \int_{1-x}^{1-c} \frac{|g(t)|^k}{t^{2k}} \, dt$$

$$= K \int_0^{1} (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)} \right)^k dx \int_{1-x}^{1-c} \frac{G^k(t)}{t^{2k+1}} \, dt$$

$$= K \int_0^{1-c} \frac{G^k(t)}{t^{2k+1}} \, dt \int_{1-t}^{1} (1-x)^{3k-1} \left(\frac{p''(x)}{p(x)} \right)^k dx$$

$$\leq K,$$

by the hypothesis (7.3.1) .

Finally,

(7.5.6)
$$I_{13} = \frac{1}{\pi}\int_{1-c}^{\pi}|g(t)|^k dt \int_0^1 (1-x)^{k-1}\left|P_1'(x,t)\right|^k dx$$

$$\leq \frac{K}{\pi}\int_{1-c}^{\pi}|g(t)|^k dt \int_0^1 (1-x)^{3k-1}\left(\frac{p''(x)}{p(x)}\right)^k dx$$

$$\leq K\{G(c)\}^k \leq K ,$$

again by the hypothesis of the theorem.

Combining the inequalities (7.5.4), (7.5.5) and (7.5.6), we get

$$I_1 \leq K .$$

Similarly, I_2 is also finite and this completes the proof of our theorem.

CHAPTER VIII

[illegible] INVOLVING
[illegible]

8.1 <u>Definitions and Notations</u>. Let $\{s_n\}$ be a sequence of real numbers. The sequence-to-sequence Quasi-Hausdorff transform,[1] or simply the (H^*, χ)-transform, h_n^*, of the sequence $\{s_n\}$ is defined by :

$$(8.1.1) \qquad h_n^* = \sum_{k=n}^{\infty} \int_0^1 s_k \binom{k}{n} t^{n+1}(1-t)^{k-n} \, d\chi(t), \quad (n=0,1,2\ldots),$$

where $\chi(t)$ is a function of bounded variation in the closed interval $[0, 1]$.

The summability (H^*, χ) of the sequence $\{s_n\}$, to the sum s, is defined as the convergence to a finite limit s of its (H^*, χ)-transform, h_n^* . It is well-known [2] that the necessary and sufficient condition that the (H^*, χ)-transform be regular is that

$$(8.1.2) \qquad \chi(1) - \chi(0) = 1.$$

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1) Hardy (1), page 277.

2) Ramanujan (1), (3), (4). He discussed this method in detail. See also Hardy (1), page 278, 279.

We say that the sequence $\{s_n\}$ is absolutely summable (H^*, χ), or simply summable $|H^*, \chi|$, if $\{h_n^*\} \in BV$.

BORWEIN [1] has recently defined the following logarithmic method of summability, denoted by (L):

If

$$\ell(x) = \ell_s(x) = \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

tends to a finite limit s as $x \to 1-0$ in the open interval $(0, 1)$, we say that the sequence $\{s_n\}$ is summable (L) to the sum s. Analogously, we say that the sequence $\{s_n\}$ is absolutely summable (L), or simply summable $|L|$, if

$$\ell(x) \in BV\ (0, 1).$$

We use the following notations in the sequel:-

$$f(x) = \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1} \; ; \quad \varphi(y) = f\left(1-\frac{1}{y}\right)$$

$$f^*(x) = \sum_{n=0}^{\infty} \frac{h_n^*}{n+1} x^{n+1} \; ; \quad \varphi^*(y) = f^*\left(1-\frac{1}{y}\right)$$

$$\ell(x) = -\frac{f(x)}{\log(1-x)} \; ; \; L(y) = \ell\left(1-\frac{1}{y}\right) = \frac{\varphi(y)}{\log y}$$

---

1) Borwein (3).

$$\ell^*(x) = -\frac{f^*(x)}{\log(1-x)} \; ; \; L^*(y) = \ell^*(1-\frac{1}{y}) = \frac{f^*(y)}{\log y} \; .$$

8.9 __Introduction.__ Let A and B be two summability methods for sequences $\{s_n\}$, and let us denote by AB the iteration-product which associates with any given sequence the A-transform of its B-transform (of course, provided it is possible to define it).

Recently, for absolute summability methods, PATI and RAMANUJAN [1] raised the question : under what circumstances $|A| \subseteq |AB|$? , similar to that raised by SZÁSZ [2] for ordinary inclusion : $(A) \subseteq (AB)$, and this question has been answered by them in the affirmative for various pairs (A, B), of summability methods.

Concerning the product of summability methods, ISHIGURO [3] has recently established the following :

__Theorem A. Let (H^*, χ) be a regular quasi-Hausdorff method. If $\{s_n\}$ is bounded and__

(8.9.1) $$\int_0^{\sigma} \log t \; |d\chi(t)|$$

---

1) Pati and Ramanujan (1) .

2) Szász (1) .

3) Ishiguro (1).

is finite for a positive $\sigma(<1)$, then $L \subseteq L.(H^*, \chi)$.

The object of the present chapter is to establish an analogous result for absolute summability .

8.3 We establish the following theorems:

__Theorem 1. Let__ (H^*, χ) __be a regular quasi-Hausdorff method. If the sequence__ $\{s_n\}$ __is bounded and if__

$$(8.3.1) \qquad \int_0^1 \log t \, |d\chi(t)|$$

__is finite, then__ $|L| \subseteq |L.(H^*, \chi)|$.

__Theorem 2. Let__ (H^*, χ) __be a regular quasi-Hausdorff method. If the sequence__ $\{s_n\}$ __is bounded and if__

$$(8.3.2) \qquad \int_0^{\sigma} \log t \, |d\chi(t)|$$

__is finite for a positive__ $\sigma(<1)$, __then__ $|L| \subseteq |L.(H^*, \chi)|$.

8.4 __Proof of Theorem 1.__ For the proof, we use the method of RAMANUJAN [1] and PATI and RAMANUJAN.[2] Since the quasi-Hausdorff transform of $\{s_n\}$ is given by (8.1.1),

---

1) Ramanujan (2) .

2) Pati and Ramanujan (1) .

we have

$$(6.4.1)\quad \sum_{n=0}^{\infty} \frac{h_n^*}{n+1}\, x^{n+1} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \sum_{k=n}^{\infty} \int_0^1 s_k \binom{k}{n} t^{n+1}(1-t)^{k-n}\, d\chi(t),$$

provided the right-hand member exists.

To prove this existence, we consider the right-hand member with s_k replaced by $|s_k|$ and $\chi(t)$ supposed to be monotonic increasing (as is permissible). The right-hand member with these changes becomes

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \sum_{k=n}^{\infty} \int_0^1 |s_k| \binom{k}{n} (1-t)^{k-n}\, t^{n+1}\, d\chi(t)$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \int_0^1 \sum_{k=n}^{\infty} |s_k| \binom{k}{n} (1-t)^{k-n}\, t^{n+1}\, d\chi(t)$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \sum_{k=n}^{\infty} |s_k| \binom{k}{n} (1-t)^{k-n}\, t^{n+1}\, d\chi(t)$$

$$= \int_0^1 \sum_{k=0}^{\infty} |s_k| \sum_{n=0}^{k} \binom{k}{n} (1-t)^{k-n}\, \frac{x^{n+1}}{n+1}\, t^{n+1}\, d\chi(t)$$

$$= \int_0^1 \sum_{k=0}^{\infty} |s_k|\, t \int_0^x (1-t+ut)^k\, du\, d\chi(t)$$

$$= \int_0^1 \int_0^x \sum_{k=0}^{\infty} |s_k|\, t\, (1-t+ut)^k\, du\, d\chi(t)\,.$$

every inversion of operations being justified by the fact that we have only positive integrands or terms. The last integral is finite from the boundedness of $\{s_n\}$ and the condition (8.1.2).[1]

Hence, after ISHIGURO,[2] we get

$$(8.4.2) \quad \sum_{n=0}^{\infty} \frac{h_n^*}{n+1} x^{n+1} = \int_0^1 \int_0^x \sum_{k=0}^{\infty} s_k (1-t+ut)^k \, du \; t \, d\chi(t).$$

Since

$$\int_0^x \sum_{k=0}^{\infty} s_k (1-t+ut)^k \, du = \int_0^x f'(1-t+ut) \, du$$

$$= \frac{1}{t} \left\{ f(1-t+xt) - f(1-t) \right\},$$

using the notations of Section 8.1, we get

$$(8.4.3) \qquad f^*(x) = \int_0^1 \left\{ f(1-t+xt) - f(1-t) \right\} d\chi(t).$$

Substituting $x = 1 - \frac{1}{y}$ in (8.4.3), we obtain

$$(8.4.4) \qquad L^*(y) = \int_0^1 \frac{F(y/t)}{\log y} \, d\chi(t) - \int_0^1 \frac{f(1-t)}{\log y} \, d\chi(t)$$

---

1) see Ishiguro (1) .

2) Ishiguro (1), relation (5) .

$$= \int_0^1 L(y/t)\,\frac{\log(y/t)}{\log y}\,d\chi(t) + \int_0^1 \frac{\log t}{\log y}\,d\chi(t)$$

$$= \int_0^1 L(y/t)\,d\chi(t) - \int_0^1 L(y/t)\,\frac{\log t}{\log y}\,d\chi(t) +$$

$$+ \int_0^1 \frac{\log t}{\log y}\,d\chi(t)$$

$$= L_1^*(y) - L_2^*(y) + L_3^*(y), \text{ say.}$$

Thus, by hypothesis, $L(y) \in BV[1, \infty)$, and we need to show that $L^*(y) \in BV[1, \infty)$, i.e., it is enough to show that

$$L_r^*(y) \in BV[1, \infty), \quad (r = 1, 2, 3) .$$

Since $L(y) \in BV[1, X]$ for each X, given an arbitrarily small positive number ϵ, there exists a number y', such that var $L(y) \leq \epsilon$, for $y \geq y'$, that is, in each interval $[y', X]$, var $L'(y) < \epsilon$.

Let us fix $\delta > 0$ such that $y'\delta = 1$; evidently $0 < \delta < 1$.

Let $y \in [1, X]$, $X > 1$ arbitrary. Then, for t in $[0, \delta]$, we have $y/t \geq y/\delta \geq 1/\delta = y'$ and therefore

$$\mathrm{var}_y \int_0^{\delta} L(y/t)\,d\chi(t) \leq M ,$$

where N is finite, independent of δ and in fact as small as we please, using the fact that $\chi(t) \in BV\,[\delta, 1]$. Also, for t in $[\delta, 1]$ and y in $[1, X]$, we have $1 \leq y/t \leq y/\delta = X$, say, and therefore

$$\mathrm{var}_y \int_{\delta}^{1} L(y/t)\, d\chi(t) \leq K ,$$

where K is finite, independent of δ, since $L(y/t) \in BV\,[1,X]$ for each X and $\chi(t) \in BV\,[0, 1]$. Thus

$$L_1^*(y) = \int_0^1 L(y/t)\, d\chi(t) \in BV\,[1,\infty) .$$

Similarly, we can prove that

$$L_2^*(y) = \int_0^1 \frac{L(y/t)}{\log y} \log t \; d\chi(t) \in BV\,[1,\infty) ,$$ since $L(y/t) \in BV\,[1,\infty)$, $\frac{1}{\log y} \in BV\,(1,\infty)$, and the fact that

$$\int_0^1 \log t \,\left| d\chi(t) \right|$$

is finite.

Again, for $y \in [1,X]$, where X is arbitrary, and any arbitrary partition $1 = y_0 < y_1 < \ldots < y_n = X$ of $[1,X]$, we have

$$\left| L_3^*(y_i) - L_3^*(y_{i-1}) \right| = \left| \int_0^1 \left[\frac{1}{\log y_i} - \frac{1}{\log y_{i-1}} \right] \log t \; d\chi(t) \right|$$

$$\leq \int_0^1 \left| \frac{\log y_{i-1} - \log y_i}{\log y_i \log y_{i-1}} \right| \log t \left| d\chi(t) \right| .$$

so that

$$\operatorname{var}_y L_3^*(y) = \sum_{i=1}^{n} \left| L_3(y_i) - L_3(y_{i-1}) \right|$$

$$\leq \int_0^1 \left\{ \sum_{i=1}^{n} \left| \frac{1}{y_i \log y_i \log y_{i-1}} \right| \right\} \log t \left| d\chi(t) \right|$$

$$\leq K \int_0^1 \log t \left| d\chi(t) \right|$$

$$\leq K ,$$

by hypothesis. Thus $L_3^*(y) \in BV\,[1,\infty)$.

This completes the proof of the theorem.

8.5 Proof of Theorem 2. We see that

$$\int_0^1 \log t \left| d\chi(t) \right|$$

$$= \int_0^{\sigma} \log t \, |d\,\chi(t)| + \int_{\sigma}^{1} \log t \, |d\,\chi(t)|$$

$$= O(1) + \lim_{\epsilon \to 0} \int_{\sigma}^{1-\epsilon} \log t \, |d\,\chi(t)|$$

$$= O(1) + \lim_{\epsilon \to 0} \log (1 - \epsilon) \int_{\sigma}^{1-\epsilon} |d\,\chi(t)|$$

$$= O(1) + o(1) = O(1) ,$$

by the fact that

$$\int_{\sigma}^{1} |d\,\chi(t)| \to 0 , \quad \text{as } \sigma \to 1 ,$$

since $\chi(t) \in BV\,(0, 1)$.

Hence, Theorem 2 follows immediately from Theorem 1.

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APPENDIX

The Aligarh Bulletin of Mathematics
Department of Mathematics and Statistics
Aligarh Muslim University, Aligarh.

Dated : Oct.1[illegible], 1974

Dear Sir,

I am glad to inform you that your article entitled 'On $|J, p_n|_k$ summability of Fourier Series II', has been accepted for publication in our bulletin.

It will be published in one of the forthcoming issues.

Sd/- Z.U. Ahmad
Managing Editor

Mr. [illegible] Abdul Rahiman,
Department of Maths and Stats
A.M.U. Aligarh.

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